

Philosophical Thought in Ancient Chinese Mathematics-- A Study of Confucianism in The Nine Chapters on the Mathematical Art and Liu Hui's Commentary

Kewei Li

Keystone Academy, Beijing, China

Abstract. This paper examines the ontological and epistemological ideas found in *The Nine Chapters on the Mathematical Art* (*Jiuzhang suanshu*) and *Liu Hui's Commentary*, clarifying the fundamental principle of "action precedes knowledge" in ancient Chinese mathematical thought from the perspective of the philosophy of mathematics. The study further compares this Chinese mode of thinking with Western mathematical ontology, focusing on methods explored in *Liu Hui's Commentary*, such as the "Jinyou Rule" and Liu Hui's π algorithm. These approaches are contrasted with Western methods, including Archimedes' "double reductio ad absurdum," to illuminate the unique role of "action precedes knowledge" within Chinese ontological philosophy. Through this comparison, the paper highlights the distinct philosophical foundations underlying Eastern and Western approaches to mathematics.

Keywords: Liu Hui; *The Nine Chapters on the Mathematical Art*; Ontology of Mathematics; Chinese Philosophy of Mathematics; Confucianism.

1. Introduction

Hegel pointed out that truth is constituted by the relationships among all phenomena. This fundamental way of thinking aligns closely with core principles in Chinese philosophy, such as "ontology embedded in things" and "action precedes knowledge", particularly in their emphasis on the importance of phenomena. However, it also diverges fundamentally in the order of priority between truth and knowledge. Through an exploration of the ontological and epistemological ideas in *The Nine* and *Liu Hui's Commentary*, this research seeks to clarify the foundational principles of ancient Chinese mathematical thought from the perspective of the philosophy of mathematics, examining how these principles contribute to and expand upon Hegel's dialectical thought.

This study offers a preliminary philosophical analysis of the inferential and dialectical characteristics in *The Nine* and *Liu Hui's Commentary*, focusing on the ontological idea of "action precedes knowledge" and its contrast with the Platonic model of "concept and deduction." The research highlights Liu Hui's mathematical reasoning methods, such as the Jinyou Rule and Liu Hui's π algorithm, and contrasts these with Western methods, including Archimedes' "double reductio ad absurdum," thereby illustrating the unique role of "action precedes knowledge" in the context of Chinese philosophy.

Furthermore, within the broader framework of global intellectual history and scientific inquiry, the comparison of *The Nine Chapters on the Mathematical Art* and *Liu Hui's Commentary* with Euclid's *Elements* reveals commonalities in the thought processes of early mathematicians from the East and the West. At the same time, it highlights distinctly Chinese elements in Liu Hui's work, demonstrating how scholars from diverse cultural backgrounds, while investigating similar natural, logical, and scientific phenomena, chose to emphasize different aspects. This study also shows how scholars' research methods were profoundly influenced by the prevailing values and societal norms of their respective cultures.

2. Research Background

Current literature on *The Nine Chapters on the Mathematical Art* follows one of four main approaches. The first is exegetical analysis, involving detailed interpretation of ancient manuscripts

to uncover linguistic and conceptual nuances. The second approach, mathematical reconstruction, has scholars translate *Liu Hui's Commentary* into modern mathematical language to clarify and contextualize his deductive methods. A third approach, philosophical inquiry, examines the underlying philosophical principles within *The Nine Chapters on the Mathematical Art* and *Liu Hui's Commentary*, exploring them through the lens of Neo-Confucianism and traditional Chinese philosophy of substance and function. Lastly, a comparative study approach places *The Nine Chapters on the Mathematical Art* and *Liu Hui's Commentary* Alongside Pre-Renaissance Western mathematical works, analyzing their methodology, deductive processes, and textual structure to highlight cultural and intellectual contrasts.

In the first two methods, exegetical interpretation and mathematical reconstruction, scholars like Zhenwei Xi and Qincong Zhang have conducted in-depth analyses of traditional mathematical terms, such as "excess" and "ratio", using modern algebraic language to enhance understanding. The third method, philosophical inquiry, is represented by Meixia Zhao and Xinghui Lei's work, *Confucian Thought and Liu Hui's Commentary on The Nine Chapters on the Mathematical Art*, which was among the first to reconstruct Liu Hui's reasoning model and deductive principles by applying universal principles of logic and formal deduction. However, their study did not further delve into Liu Hui's ontological philosophy. Specifically, Zhao and Lei propose that the Confucian idea of "accordance and verification" shaped Liu Hui's rigorous scholarly approach, though they did not realize that this principle of verification is more than a broad, abstract academic style; rather, it serves as the foundational source for the core philosophical idea in *The Nine Chapters on the Mathematical Art* and *Liu Hui's Commentary* that "ontology is embedded in things".

In contrast, Jizhen Hu, Zhifang Zhu, and Dianshun Hu's article, *Preliminary Exploration of Ontology in Traditional Chinese Mathematical Philosophy*, acknowledges the influence of the verification principle on Liu Hui and recognizes an early reflection of the Confucian concept of "investigation of things to attain knowledge" in *Liu Hui's Commentary*. They notably argue, with originality, that the traditional Chinese epistemological view of "action precedes knowledge" fundamentally shaped Liu Hui's mathematical ontology, leading him to embed "number" within the material world.

This ontological perspective, which unifies "number" with the material world, stands in stark contrast to the dualistic ontology of the Platonic tradition, where "forms" are considered primary to physical objects. Xi and Zhang's research further highlights that this foundational difference is what enabled *Liu Hui's Commentary* to address the "fear of infinity" that had troubled Archimedes.

3. Research Question

This study presents a preliminary philosophical analysis of the inferential and dialectical characteristics in *The Nine Chapters on the Mathematical Art* and *Liu Hui's Commentary*, which are among the most representative achievements in the history of ancient Chinese mathematical thought. It focuses on the ontological ideas expressed in *The Nine Chapters on the Mathematical Art* and *Liu Hui's Commentary*, such as "action precedes knowledge", "investigation of things to attain knowledge", and the concept of "ontology embedded in things". The study also examines the differences in thought patterns between this approach and the Platonic style of "concept and deduction logic" that characterizes Western mathematical ontology. The research examines key mathematical methods in *Liu Hui's Commentary*, including the Jinyou Rule and Liu Hui's π algorithm, and contrasts these with Western mathematical approaches, such as Archimedes' "double reductio ad absurdum."

4. Methodology

This study employs a combination of literature review and practical mathematical calculation. The primary focus is on conducting a textual comparison between the ancient text of *The Nine Chapters*

on the *Mathematical Art* and contemporary philosophical research on *Liu Hui's Commentary*. Through computational examples, the study demonstrates how Liu Hui's mathematical reasoning elaborates on and reinforces the mathematical-philosophical principles within *The Nine Chapters on the Mathematical Art*. Additionally, by integrating philosophical commentary with mathematical calculations in parallel, this research reconstructs and contrasts the differing approaches taken by Archimedes and Liu Hui in solving similar mathematical problems. This comparison aims to reveal fundamental ontological differences between early Eastern and Western mathematicians, particularly highlighting how *Liu Hui's Commentary* addresses the "fear of infinity" encountered in *Elements* and other Western mathematical classics. Furthermore, it explores the divergence between Chinese ontological thought, grounded in the concept of "action preceding knowledge" and Western conceptual ontology.

5. Significance

This study introduces two distinctive research directions and methodologies in the field. First, it employs an approach commonly used in contemporary philosophy of mathematics, where actual calculations and philosophical discussions are conducted side by side to offer mutual insights. Using methods from analytical philosophy, this approach will reconstruct key results from *The Nine Chapters on the Mathematical Art* and *Liu Hui's Commentary*.

Second, rather than treating *The Nine Chapters on the Mathematical Art* and *Liu Hui's Commentary* as isolated texts, the study situates them within the broader context of global mathematical history. It also includes a comparative analysis with other foundational works, such as *Elements* and *Sunzi Suanjing*, to underscore the unique concept of "ontology embedded in things" found in *The Nine Chapters on the Mathematical Art* and *Liu Hui's Commentary*. Likewise, the philosophical implications of *The Nine Chapters on the Mathematical Art* are reconstructed as an integrated part of Chinese intellectual history, with particular emphasis on the principle of "action preceding knowledge" within Chinese ontological thought. This approach seeks to clearly illuminate the distinct position of this concept within the ontological framework of Chinese philosophy.

6. Background and Structure of *The Nine Chapters on the Mathematical Art*

As one of the oldest surviving mathematical texts in China, *The Nine Chapters on the Mathematical Art* stands out for its remarkable achievements, which are attributed to the systematic thinking of its authors and their practical application of the Confucian principle of "action precedes knowledge". *The Nine Chapters on the Mathematical Art* compiles mathematical methods and principles from the Zhou dynasty onward, presenting mathematical thought in a comprehensive and structured way that encompasses a broad spectrum of mathematical theories [1]. Consequently, by studying the dialectical approaches and mathematical accomplishments within *The Nine Chapters*, scholars can gain deeper insights into the intellectual methods and scientific thinking of the Chinese people during specific historical periods, thereby enhancing the understanding of Confucian philosophy and Chinese cultural heritage.

In the Northern Song dynasty, the Chinese mathematician and astronomer Liu Hui provided detailed annotations and explanations for *The Nine Chapters on the Mathematical Art*, resulting in *Liu Hui's Commentary*. His annotations organize and clarify each chapter, adding example problems and practical applications, which make *The Nine Chapters on the Mathematical Art* more comprehensive and easier to understand, leaving a lasting impact on subsequent mathematical research and pedagogy. *The Nine Chapters on the Mathematical Art* and *Liu Hui's Commentary* laid the foundation for the structure, format, style, and defining characteristics of ancient Chinese mathematics, and they pioneered a deductive approach where necessary conclusions are drawn based on fundamental concepts and theories.

Liu Hui's Commentary is distinguished by its algorithmic and model-based approach, exemplified through a consistent structure across its explanations, which comprise three parts: problem, solution, and explanation. The "problem" and "solution" represent the question and its answer, while the "explanation" elaborates on the mathematical method used to solve the problem [2]. The algorithms in *Liu Hui's Commentary* feature detailed steps, transforming "techniques" into mathematical models where results can be derived using counting rods, demonstrating high operability and practicality [3]. Furthermore, the text reveals close connections between different methods. For example, in the Chapter on Figuring for Construction (商功章), spheres are positioned relative to corresponding cuboids, while the "rectangular order arrangement" and the "common techniques for curves and straight lines" for prisms with rectangular bases reflect an internal coherence grounded in the principle of cross-sections.

Liu Hui's Commentary also refines *The Nine Chapters on the Mathematical Art* by choosing a new logical starting point for its theoretical framework, reconstructing the basic properties of "ratios", which the original text applied narrowly. By establishing ratios as a core concept, Liu Hui derives further theories on positive and negative numbers, area, and volume [4]. Consequently, both *The Nine Chapters on the Mathematical Art* and *Liu Hui's Commentary* exhibit a holistic and systematic approach to presenting a functional mathematical theoretical system. This comprehensive and organized method reflects the Confucian scholar's ideal of unifying all categories under a single guiding principle, while also endowing *Liu Hui's Commentary* with a rigor that can withstand scrutiny by modern mathematical scholars.

7. "Action Precedes Knowledge": Using the Jinyou Rule and Liu Hui's π Algorithm as Examples

7.1. Liu Hui's Jinyou Rule

Liu Hui's principle of "guiding mathematical methods with established rules" exemplifies the Confucian concept of "action precedes knowledge" in *Liu Hui's Commentary*. Specifically, Liu Hui often begins with an understanding of real-world problems and specific contexts, then designs solutions based on existing principles, such as social norms and universal values. For instance, the Jinyou Rule presents a mathematical approach for solving the problem of "equal distribution of millet," which addresses issues related to the fair allocation of resources.

Suppose county A has 10,000 households and takes 8 days to travel. County B 9,500 households and takes 10 days to travel. County C has 12,350 households, and takes 13 days to travel. County D has 12,200 households and 20 days to travel. In total they send 250,000 units of food with 10,000 carts. Please calculate each county's number of units of food transported, and the number of carts used.

The Nine Chapters on the Mathematical Art and *Liu Hui's Commentary* take into account the number of households in each county and the number of travel days required. By dividing the number of households in each county by the travel days, a "declining ratio" is established, meaning that each household contributes one cart for one day of travel. This ratio represents the proportional share assigned to each household. The sum of these ratios is then divided by the total number of carts needed to transport the grain, determining the proportion of carts each county should contribute. Finally, multiplying this result by 25 hu (a unit of volume) yields the total amount of grain each county is responsible for transporting [5].

Liu Hui's approach presupposes a foundational understanding of the concept of "distribution by proportion", as reflected in the setup, resolution, and analysis of the problem. This understanding extends beyond what can be derived solely from mathematical theories and algorithms. Indeed, alternative solutions to the problem could include assigning all transport responsibilities to County A, thereby minimizing travel time and maximizing efficiency. However, in contrast to Liu Hui's method, which incorporates the principle of "equitable distribution", such an approach would lack fairness.

Liu Hui instead applied a principle of proportional contribution, assigning transportation tasks so that each household contributes according to a proportional share, thereby ensuring fairness in effort. This methodology exemplifies his guiding principle that "established principles direct mathematical methods," further illustrated in the context of "distribution by proportion."

Suppose County A has 1,200 people and is near the frontline. County B has 1,550 people with a travel time of 1 day; County C has 1,280 people with a travel time of 2 days; County D has 990 people with a travel time of 3 days; and County E has 1,750 people with a travel time of 5 days. The five counties collectively need to send 1,200 soldiers for a one-month service period. Calculate the number of soldiers each county should send.

Using the "Jinyou Rule" provided by Liu Hui in the previous example:

Listing the Declining Ratios Divide the number of soldiers each county should send by the sum of the service duration (1 month, counted as 30 days) and the travel time. This gives a proportional ratio, known as the "declining ratio."

$$\begin{aligned} \text{County A: } \frac{1200}{30} &= 40 \\ \text{County B: } \frac{1550}{31} &= 50 \\ \text{County C: } \frac{1280}{32} &= 40 \\ \text{County D: } \frac{990}{33} &= 30 \\ \text{County E: } \frac{1750}{35} &= 50 \end{aligned} \tag{1}$$

The declining ratio represents the draft proportion in each county. For example, a declining ratio of 30 for County A means that one soldier is drafted for every 30 people, while a declining ratio of 31 for County B means that one soldier is drafted for every 31 people, and so on.

Simplify For ease of calculation, Liu Hui divides all the declining ratios by their greatest common divisor (GCD), which in this case is ten, which maintains the original proportions.

$$\begin{aligned} \text{County A: } 4 \\ \text{County B: } 5 \\ \text{County C: } 4 \\ \text{County D: } 3 \\ \text{County E: } 5 \end{aligned} \tag{2}$$

Multiplying Each Multiplying each declining ratio by 1200, the number of soldiers needed:

$$\begin{aligned} \text{County A: } 4 \times 1200 &= 4800 \\ \text{County B: } 5 \times 1200 &= 6000 \\ \text{County C: } 4 \times 1200 &= 4800 \\ \text{County D: } 3 \times 1200 &= 3600 \\ \text{County E: } 5 \times 1200 &= 6000 \end{aligned} \tag{3}$$

Dividing Each Dividing each by the sum of the declining ratios:

$$\begin{aligned} \text{County A: } 228\frac{4}{7} \\ \text{County B: } 285\frac{5}{7} \end{aligned}$$

$$\text{County C: } 228\frac{4}{7} \quad (4)$$

$$\text{County D: } 171\frac{3}{7}$$

$$\text{County E: } 285\frac{5}{7}$$

$$\text{Total: } 1200$$

In this step, Liu Hui reduced the results of “multiplying each” proportionally, and gets the actual number of soldiers each county should send. The total number of soldiers sent is 1,200, as we need.

Number of Soldiers Sent

$$\text{County A: } 229$$

$$\text{County B: } 286$$

$$\text{County C: } 228 \quad (5)$$

$$\text{County D: } 171$$

$$\text{County E: } 286$$

$$\text{Total: } 1200$$

Finally, Liu Hui rounded the decimal results to provide a specific, actionable plan [6]. The concept of "action preceding knowledge" is reflected in Liu Hui's incorporation of social fairness—a pre-existing societal norm—into mathematical problems. His "action" in social life guided his "knowledge" in mathematics, demonstrating how practical societal concerns shaped his mathematical reasoning [7].

7.2. Liu Hui's π algorithm

Additionally, Liu Hui's exploration of the concepts of infinity and the infinite was also guided by the principle of "action preceding knowledge" and demonstrates the influence of "accordance verification" within his Confucian-based mathematical framework.

For instance, when calculating the area of a circular field, Liu Hui used the concepts of infinity and the infinite to develop the circle-cutting method, allowing him to approximate a value very close to the actual area. Suppose there is a circular field with a circumference of 30 steps and a diameter of 10 steps, and we want to find its area. Liu Hui inscribed regular polygons within the circle, making the area of these polygons approach the area of the circle with increasing accuracy. By calculating the areas of an inscribed hexagon and dodecagon, he demonstrated that the circle's area is greater than that of the dodecagon. He then inscribed a rectangle along one side of the hexagon, overlapping with the dodecagon. The area of this rectangle represents the difference between the areas of the dodecagon and the hexagon [8]. Through this approach, Liu Hui was able to approximate the area of the circle by determining a range based on these inscribed shapes.

$$S_{12} < S_{\text{circle}} < 2S_{12} - S_6 \quad (6)$$

Transforming to a universal formula:

$$S_{2n} < S_{\text{circle}} < 2S_{2n} - S_n (n \geq 3) \quad (7)$$

"The arc field diagram allows a square to contain a circle, and within the circle, another square, where the inner square is half the area of the outer square." This refers to the area of an inscribed square within a circle being half the area of the circumscribed square. As shown in the diagram below:

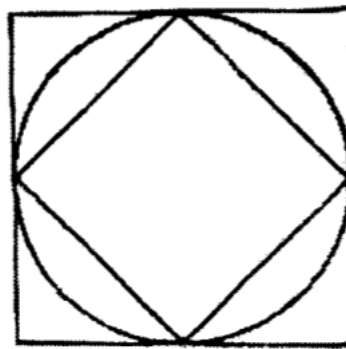


Fig. 1 The circle with an outer circumscribed square and an inner inscribed square

(Adapted from Guo Shuchun's annotated translation of *The Nine Chapters on the Mathematical Art*)

"Given a circumference of 157 and a diameter of 50, we derive the ratio between them. This ratio is still slightly less than the actual value [9]."

Thus, we calculate π as: $\pi \approx 157/50$. (8)

Using Liu Hui's ratio, the area of the circular field is:

$$S_{\text{circle}} = 71 \frac{103}{157} (\text{square}) \text{steps} \quad (9)$$

Unlike Liu Hui's approach, Archimedes' "double reductio ad absurdum" method for determining the area of a circle did not initially involve the concept of infinity. Instead, he began by estimating a range and progressively proving that the area of a circle could neither exceed a certain value nor fall below another [10], thus addressing the area problem. In contrast to Liu Hui, Archimedes clearly developed an abstract method before applying it to practical scenarios. His "double reductio ad absurdum" approach relied entirely on prior estimates and rough calculations, whereas Liu Hui's π algorithm was a constructive, practice-based approach, with each step grounded in the construction of an actual geometric figure.

This contrast highlights that, for Archimedes, "knowledge" preceded "action", while Liu Hui prioritized "accordance" through practical application, confirming "validation" through theory. As Xunzi observed, "In any discourse, what matters is achieving consistency and verification." The concept of "accordance and verification" underpinned the close connection between *The Nine Chapters on the Mathematical Art* and *Liu Hui's Commentary* with practical application, aligning seamlessly with the Confucian ideal of "learning for practical application," which values theory's relevance to real-world use [11]. This distinction—between Liu Hui's practice-based "accordance and verification" and Archimedes' abstract, theory-first approach—also underscores the fundamental cultural differences between East and West, reflecting the distinct intellectual landscapes of their respective eras.

8. Conclusion

Current literature on *The Nine Chapters on the Mathematical Art* primarily focuses on philological and classical studies, the reconstruction of deductive processes, exploration of underlying philosophical principles, and comparative analysis with Western mathematics. Although Meixia Zhao and Xinghui Lei reconstructed the reasoning model in *Liu Hui's Commentary*, they did not delve deeply into Liu Hui's core ontological ideas. Conversely, in their study *A Preliminary Exploration of Ontology in Traditional Chinese Mathematical Philosophy*, Jizhen Hu, Zhifang Zhu, and Dianshun Hu innovatively proposed that the Chinese epistemological concept of "action preceding knowledge" shaped Liu Hui's mathematical ontology, inherently situating "numbers" within "things." Xi and

Zhang also pointed out that this fundamental difference enabled *Liu Hui's Commentary* to address Archimedes' "fear of infinity."

Unlike previous research, this study employs an approach that combines actual calculations with philosophical discussion, integrating mathematics and philosophy to reconstruct key results from *The Nine Chapters on the Mathematical Art* and *Liu Hui's Commentary*. It analyzes how core social-philosophical distinctions influenced different mathematical approaches to problem-solving. At the same time, *The Nine Chapters on the Mathematical Art* and *Liu Hui's Commentary* are not discussed as isolated texts; rather, they are examined through cross-cultural and cross-regional comparisons with *Elements*, exploring how diverse social and cultural contexts shaped varying methods for addressing similar mathematical problems.

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