Reflection on Set theory: Is the barber example a genuine illustration of Russell's paradox?

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Abstract. Since the formulation of Russell's paradox, many people have created more accessible models while trying to understand and solve the paradox. The Barber paradox is the most famous one, but it is not the case that this paradox was not proposed by Russell. This paper will demonstrate the nature of both paradoxes through truth-functional language and propose possible solutions (theory of types) for Russell's paradox. The reason why the Barber paradox is a pseudo paradox will also be illustrated with a possible solution. There is a huge difference between the paradoxes due to the fundamental difference in the set, and more reasons will be clarified in the paper.

Keywords: Russell's Paradox; Barber Paradox; Degrees of paradoxicality.

1. Introduction

Russell's paradox is also known as the third crisis in the history of mathematics. The crises before are caused by the inexact use of calculus. Mathematicians like Niels Henrik Abel then come to axiomatize the system of calculus, at the same time, Georg Cantor established an axiomatic system in *Crelle's Journal*, which is used to represent all the mathematical propositions. Cantor proved the well-known theorem in the 18^{th} century that the set of real algebraic numbers can be put in one-to-one correspondence with the set of natural numbers, whereas the same is not true of the set of real numbers. However, this result leads to the discovery of infinities and the different sizes of infinity. Set theory, later proposed as more advanced by Frege, is one of the most fundamental systems in mathematics.

Barber's paradox is often seen as of equal value as Russell's paradox and as an illustration of it. This paper will discover the solution to both paradoxes and scrutinize the relationship between them. According to Quine, the paradox is a 'successful having as its conclusion and statement or proposition that seems false and absurd' [1]. This paper will prove in later sections why Barber's paradox should not be paradox by a temporal solution. The problem of Russell's paradox was raised in the early 19th century, between the letter of Russell and Frege, Russell questioned Frege's idea that a function, too, can act as the indeterminate element' doubtful [2]. Then he throws this question if *w* is the predicate: to be a predicate that cannot be predicated of itself. Can *w* be predicated itself? This paper will demonstrate this paradox in detail and expound on how Russell later established a new theory (theory of types) to categorize the system.

The purpose of this paper is to confirm whether the Barber paradox can illustrate Russell's paradox, and if not, what factor should be reconsidered. If the Barber paradox is different from Russell's paradox, then it should no longer represent it and continue to mislead readers. This paper will provide advice on how to formulate a representative genuine of Russell's paradox in a later chapter.

2. The nature and solutions of paradoxes

2.1 Russell's paradox

Bertrand Russell, in 1902 in a letter to Frege about Frege's new book, Begriffsschrift, mentioned his confusion about set theory, which overturned the general belief that set theory could represent all mathematical propositions [2]. Russell proposed a plausible concept 'the set of all sets that do not contain themselves. Russell attacks the generalization principle by Cantor, where if set A contains all elements for property P, then for any property P, A={x|P(x)}. This paper believes that Russell's paradox is a result of the vague and overly open definition of set. Taking the set A from above, If A includes itself as a member, then "the set of all sets that do not contain themselves" will not be satisfied. On the other hand, if A does not include itself as a member, it will satisfy the condition: 'the set of all sets that do not contain themselves and be in set A. This is an inconsistent statement where it is both true and false.

In the following years, many 'solutions' only tried to avoid the paradox itself through formal prohibitions against the formal characteristics of the paradox. Apart from the Zermelo-Fraenkel set theory, Russell also proposed a solution, the type of theory [3]. He attributed the paradoxes to their self-referential sentence form. Therefore, he considered that no object should contain itself.

Bertrand Russell uses hierarchical branching so that the predicates at the lowest level can be defined by this system talking only about objects, and the predicates at the next level can be defined by talking about objects and the predicates at the upper level [4]. Intuitively, the world is composed of many individual elements, known as type 0. Elements in type 0 can form sets, which are type 1. Those first-order sets can also be composed into sets, which is the set of sets; this is type 2 [5]. Type theory regulates that: $x \in y$ is meaningful only if the type of x is one level lower than the type of y, and mixing classes are not allowed. In this way, according to the division of types and this principle, it can be concluded that "$A \in A$" is meaningless.

For Russell's Paradox: $A=\{x|x\in x\}$, if element x is in type n, and the set A that includes x must be in type n+1 [6]. In this case, $A\neq x$. Otherwise, for any set A, that is not in type n+1 when x is in type n, it obeys type theory. Therefore, if set A and property P cannot be in the same type, then it is impossible to be self-contradictory.

However, many argue that branch-type theory is not a practical solution to a language transformation. The branch theories had too many restrictions on the numbers and set of numbers, which makes other mathematical conclusions inconsistent. Instead of saying how all individual functions are, people can only say how functions of a certain order are [7]. He essentially increased the difficulty of defining individuals, as the theory of types works by narrowing down the functions. Asa result, it makes many parts of elementary mathematics impossible. The incompleteness and semi-decidable feature of the theory of types cause his impracticality.

2.2 Barber's paradox

The barber paradox is an illustration of Russell's paradox, given by Bertrand Russell in 1903: A barber in the town only shaves those who do not shave [8]. Does the Barber shave himself? If he does, he is a man in the group that 'does shave themselves'. So, he cannot shave. If he does not shave himself, he is in the group that 'does not shave themselves'. Therefore, the Barber needs to shave himself.

This inconsistent argument is often used as an example or simple explanation of Russell's paradox. However, many argue that this argument can only prove the fact that this kind of Barber does not exist.

Let x and y be the people who do not shave themselves. B be the Barber and S be the action of shaving [9].

X shaves himself: $S(x,x)$

X does not shave himself: $\neg S(x,x)$

The Barber shaves someone: $S(B,x)$

The paradox can be represented by: $\neg S(x,x) \Rightarrow S(B,x)$

Hence, If the Barber does not shave himself and the paradox is true, then:

$$x=B \quad \rightarrow \quad \neg S(B,B) \Rightarrow S(B,B) \tag{1}$$

The above premise is wrong as it is a contradiction, as $\neg S(B, B)$ should not be equal to $S(B, B)$. Therefore, if x=B, then the paradox is wrong. However, when $x\neq B$, $\neg S(y, y) \Rightarrow S(B, y)$ would be correct. Using this model, we can see that the paradox itself is not valid. Barber's paradox seems similar to the liar's paradox, with a problem of self-involvement.

Unlike what many people believed in the past year that 'the barber paradox was a veridical paradox showing that there is no such barber', there is one way where Barber's paradox can be solved, by adding time into the paradox [10]. According to the figure below. There are two important times: A and B. At time A, the barber declares: All the people in the town who do not shave themselves can now come to me and I will shave them. The barber can shave all the other people in the town after time A, because the barber did not shave himself before time A. Therefore, he satisfies the condition, where he is in the group of people that 'does not shave themselves'. According to what he declared at A; he will shave all the people who satisfy the condition. He can shave himself at time B. There are no theoretical contradictions.

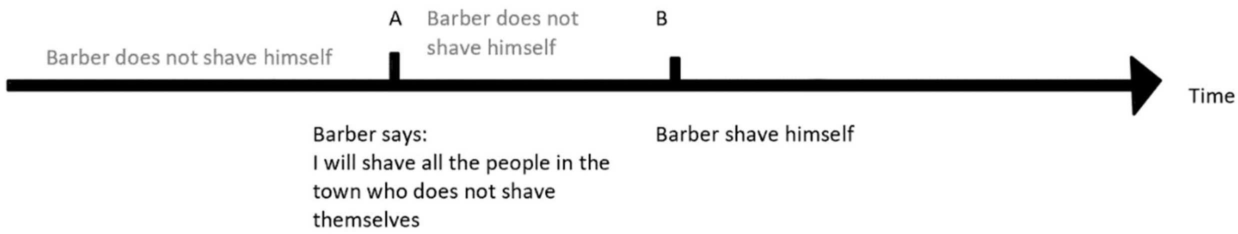

Fig. 1 diagram of the temporal solution for BP

The reason why people might think it is a paradox is because they obfuscated time. And even ignored the time condition. Following the timeline, the condition is satisfied at B, it is right for the barber to shave himself, as the barber is allowed to shave himself, the problem is solved, thence, it is not a paradox.

3. Comparison and the nature of set theory

3.1 Dissimilarities and similarities between the two paradoxes

Although many believe that Barber's paradox is the same as Russell's paradox or sees them as equal value, by comparing the two, there are still many differences between the two. The Barber's paradox is just an illustration of Russell's paradox. From the perspective of language, Barber's paradox is inherently false in the first place. The Barber's paradox only proves that no such barber exists, or the Barber said something that is never satisfactory, and both conclusions are highly acceptable. In contrast, the conclusion of Russell's paradox is not so. There is no set such that it cannot contain itself is hard to be accepted.

In contrast, Barber's paradox had more preconditions than the Russell paradox. It is also limited by time when he is limited by himself. (Even though the limitation of time did help to solve the paradox). He is not as abstract as Russell's paradox [11]. Nevertheless, it is precisely because Russell's paradox is vague and overly open so that Barber's paradox is more straightforward to tackle than Russell's paradox. Some people might see the paradoxes are of equal value, interjecting the additional premises of Barber's paradox into the reconstruction of Russell's paradox. As every property has a correlated instance of every property with the view that every predicate defines a set, and the theory of the instantiation of properties is unlikely to have adherents, Barber's paradox seems to be unconvincing.

However, they do have similarities in the structure of the settings. i.e., the set A that does not contain itself and the Barber who does not shave himself. Nevertheless, the idea that a set that does not contain itself ($X=\{x: x \notin x\}$, is different from that the Barber only shaves those who do not shave themselves $\{x: x \notin Bx\}$. We can see the essential difference between them through empty sets. No sets in Russell's paradox are specified. However, the empty set of individuals is specified in the Barber paradox. A person who shaves himself because he does not shave himself does not exist. Therefore, there are no such individuals where it is B and $\neg B$ [12].

Therefore, the two paradoxes are different intrinsically, whereas Barber's paradox is not a paradox strictly speaking. The reasons are as follows: In Russell's paradox, only the set itself is mentioned, and itself is logically valid. Barber's paradox, however, does not contain the set theory as its premises, which does not follow the concept of $x \in x$. As this paper illustrated earlier, Barber's paradox itself is not logically valid and equally valued to Russell's Paradox. Also, as it contains the word 'shave', the time factor and the verbal action of the barber allow this paradox to be solved. Hence, it is a pseudo paradox [12].

3.2 New illustration of Russell's paradox

As Barber's paradox cannot fully demonstrate Russell's paradox, this paper provides another illustration of the paradox using the following example: Although many believe that Barber's paradox is the same as Russell's paradox or sees them as equal value, by comparing the two, there are still many differences between the two. The Barber's paradox is just an illustration of Russell's paradox. From the perspective of language, Barber's paradox is inherently false in the first place. The Barber's paradox only proves that no such barber exists, or the Barber said something that is never satisfactory, and both conclusions are highly acceptable. In contrast, the conclusion of Russell's paradox is not so. There is no set such that it cannot contain itself is hard to be accepted.

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Let there be a person P. P only knows the name of the people who do not know their name. Hence, the knowledge of the names has no time limit, where P will not forget people's names or learn people's names. He has the knowledge at the very beginning.

Does P know his own name? If he does, then he should not know his name as he can 'only know the name of the people who do not know their name.' If he does not, then he should, because he is a person 'who does not know their name.' Compared to the barber example. Unlike the barber, it is restricted by the action 'shave'. The P is not restricted by an action or time; therefore, it cannot be solved by the temporal solution nor is it invalid. Therefore, the paradox illustration stands.

3.3 The effect on Set Theory and mathematics

Many pointed out that the cause of both Paradoxes is due to the self-reference. In fact, this problem of self-reference already appeared in logic before in set theory, known as the Epimenides paradox. Where Epimenides as a Cretan, had made an immoral statement where ' All Cretans are liars.' As all the work for Epimenides was lost, this paper will instead use the liar's paradox to demonstrate the idea. Russell pointed out in his book *My Philosophy Development* that only when the fact 'this sentence' in the paradox is not included in the sentence 'the sentence I said is false', we can then judge the truth or falsehood of the paradox. The liar's paradox shares the same problem as Russell's paradox. The liar's paradox was not able to be solved by Russell because it cannot be composed into more similar propositions. Although it seems to be an atomic proposition already, since it cannot be judged, it is not a proposition. From the view of predicate logic, we can see how this paradox no longer holds, as we are not considering self-reference. It is obvious to see that the Liar problem does not exist as the statement 'this sentence' is not clarified, without a clear definition of what 'this sentence' is, the Liar problem does not have a predicate, therefore, strictly speaking, this paradox does not hold. Tarski however, shared the same view as Russell. His solution to the Liar paradox is similar to Russell's theory of type. They have agreed on the existence of hierarchy in language, the order of language will naturally avoid the problem of self-referencing [13].

Russell's paradox makes all mathematicians re-examine, and set theory and self-definition. With the proposal of Russell's paradox, Mathematicians have tried many times to refine set theory. As Russell's paradox is a result of a vague definition of the set, it emphasised that the invention of 'set theory' starts with an unclear definition, which had led to more significant problems. People define sets straight as 'a collection of things' [12]. However, the 'diversity' (variables set, collections of sets, etc.)of a set directly shows that it cannot be defined this way, but can be categorized into different levels, or types. The axiom of modern set theory specifies how to build sets (the set of sets) and especially prohibits the idea of a self-definition set (a set that contains itself). By constructing subsets and hierarchies of sets, those limitations and definitions of sets naturally avoid Russell's paradox and provide a solid foundation for all modern mathematics.

Not only are the paradoxes mentioned in this article, but also many similar paradoxes are grammatically correct, it is only when we think these paradoxes can self-reference, we will fill that they are logically unreasonable. If we then by default, those issues cannot self-reference, then those question becomes incredibly easy. At the same time, self-reference itself has been avoided in general, therefore we cannot find any examples in real life which are the same as Russell's paradox. For example, we would not question when an orchard owner is trying to separate bad apples from the good ones, that the group of bad apples contains itself because we have realized it is a set of apples. The same should happen to set theory, mathematicians by building a hierarchy for set theory, let there be a set of things and a set of sets solves the problem naturally. Whereas, those theories are used to help people recognize the difference in the set, and help us to acquiesce in self-reference.

The existence of Russell's paradox does not simply imply questioning the irrationality of the definition of set theory, more importantly, it agreed with the unity between mathematics and logic. Enriched the system of axiomatic set theory. Often, these problems that seem to deny the development of mathematics, are the heroes who promote the contraction of mathematics and logic. The discovery of these paradoxes has made mathematicians aware of the logical loopholes in the concept of mathematical axioms, without them, therefore, theory of types, and other theories were established to contrast a more mature and secure mathematical axiom structure. At the same time, Russell's paradox, along with Cantor's paradox also involves the concept of infinity set, which enables mathematicians to redefine infinity [14].

4. Conclusions

This paper proposed various solutions at the same time as introducing both paradoxes and proved their similarities in many ways. Also correct that Barber Paradox is not a good genuine of Russell's paradox and strictly speaking does not illustrate Russell's Paradox. Barber's paradox is simply not logical, consistent, and valid, it might be seen that it is equally valued to Russell's paradox, however, using Truth functional language, it is proven that it is a pseudo paradox. The use of the verb 'shave' caused it to be limited by more factors such as time. Simultaneously, this also proves that Russell's paradox exists because of the conceptual inaccuracy in defining set theory. In general, the formulation of Russell's paradox has led mathematicians to redefine the set theory that was admired in the 19th century. Some people think that Russell's paradox only affects logic but does not impact mathematics as a whole. Indeed, because of its logical inaccuracies, mathematicians have perfected the concept of a set that was originally vague. Almost everyone believes that set theory is a theory that can include everything, as people accept this idea, it causes such huge crises. After Russell, and many other mathematicians such as Kurt Gödel continued their work on mathematical logic, it can be seen that paradoxes do not only show the inconsistency in logic but push force on the refinement of set theory and mathematical logic.

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