# The Set Theory and the Description of Infinity: the Nature of Infinity Perceived from Metaphysics and Ontology

Hongbo Wu[1, *]

[1] Sheffield University, South Yorkshire, United Kingdom

*Corresponding author: hwu57@sheffield.ac.uk

**Abstract.** Infinity is possibly one of the most complicated conceptions confronting mathematics and philosophy. People could always conceive it via intuition, but find it hard to describe in the logical realm. In the 19th century, German mathematician Cantor proposed the set theory, for the first time, people could use mathematical tools to explain the nature of infinity. However, such an explanation still lies on the axiomatic foundation, which does not directly fit our intuition about infinity. Therefore, this study will attempt to establish a 'bridge' between the infinity that fits intuition and the infinity that is described by set theory, hence arguing why the set theory could describe infinity universally in the senses of metaphysics and ontology.

**Keywords:** Set Theory; Infinity; Metaphysics.

## 1. Introduction

Metaphysical Infinity and Set Theory Infinity are both concepts that inspire modern science to work with a more applicable sense of the sum of entities. To start with where the foundation of the modern concept of infinity lies, It should begin with a clarification of the distinction between the infinity that fits people's intuition (Metaphysical Infinity) and the infinity that is described by set theory (Set Theory Infinity) as well as introduce the conceptions that will involve my argument in the next part. Therefore, It will present two groups of definitions. The first group will be about the definitions of infinity that people's intuition could easily fit for. The second group will be about definitions and notions that explain the set theory and its description of infinity. Hence explain why there is a distinction.

By doing so, this research will set terminology with both the definitions and characteristic descriptions for the two senses of infinity. With an interpretation scope by applying the metaphysical way of understanding infinity, this article argues that with three key characteristics, periodic, and continuity, it is possible to build a coherent structure to explain the infinity concept from the set theory with the descriptions of metaphysics and ontology.

## 2. Metaphysical Infinity

To begin with, the definitions for metaphysical infinity, suppose It will consider the conception of infinity simply based on its terminology, it means something analytically without finite, and finite only applies to something that can be quantified or abstractly quantified [1, 2]. For example, people could not say the U.S. is finite because there are no things that people specifically refer to as the subject of finite, the U.S. is not a valid subject because it can not be quantified or abstractly quantified. When someone says the U.S. is finite, typically, one might be implying something finite, such as the number of states, the population, the land size, etc., these things are valid subjects of finite because they can be somehow quantified. Some things can be abstractly quantified, for instance, people might say Superman’s power is finite because Superman’s 'power' can be abstractly quantified by comparing it with something that has higher 'power', like God.

Therefore, when people discuss infinity, is also important that it has a valid subject, infinity can not be used to describe the U.S. too, except people are implying something else that is related to the U.S. and can be used as a valid subject. Hence, a valid subject for either infinity or finite must can be quantified or abstractly quantified, then when people discuss something that can be quantified or abstractly quantified, there must be at least one object that indicates the subject can be quantified or

abstractly quantified. For example, the state of New York can be an object that indicates the federal state of the U.S. is a valid subject. Superman's power can be an object that indicates the subject of abstract quantification of power in the comparison with God. Hence, the language termed these objects that indicate a valid subject as the 'index object'.

Moving to the metaphysical forms of infinity, based on the concept of the index object, there are two typical forms of infinity that one could use the index object to categorize: Expansion Infinity and Divisional Infinity. An infinity that consists of an index object and operates 'expanding / enlargement 'based on the index object is an expansion infinity. For instance, the assertion 'the possible worlds is infinity 'is an expansion of infinity, because any possible worlds must be based on differences compared to our world and our world is the index object in this infinity. To assert the possibility of existing differences is an 'expanding' operation based on our world. An infinity that consists of an index object and operates 'division 'based on the index object is divisional. For instance, if people say time is infinity even within one second, it would be a divisional infinity because the one second is the index object, and by dividing the one-second people could get the infinite units of time. These two forms of infinity essentially cover all possible instances of infinity that one could intuitively understand or conceive. There seems to be, intuitively, no infinity that metaphysical conceivable and fits intuition can not be categorized into one of these two typical forms or both [3, 4]. But why does our intuition only possess these two ways to construct or categorized infinity? My answer is that the intuition of ourselves is bounded by the characteristics of infinity, and the characteristics of infinity only allow these two intuitive ways to construct or categorised infinity.

Moving to the characteristics of metaphysical infinity, it is worthwhile to devise suitable descriptions for this approach. There are three necessary characteristics for an object or group of objects that can be termed as an infinity while it can be metaphysical conceivable and fits intuition. These three characteristics are (1) Periodic, (2) Continuity, and (3) Unidentifiable.

Periodic is applied for any infinity to be metaphysical conceivable and fits intuition, if and only if it possessed a certain means. If there is a new object, the new object could link with the original object via such means. Continuity is applied for any infinity to be metaphysical conceivable and fits intuition, if and only if it always exists a new object. Unidentifiable can be understood as for any infinity to be metaphysical conceivable and fits intuition, if and only if those objects except the index object, are possible unidentifiable. For possible unidentifiable, suppose people ask for the 30000th number of $\pi$ after the period, it is theoretically identifiable, however, until People calculated the 30000th number of $\pi$, it will remain unidentifiable. Extending to infinity, the calculation for xth will always be possible unidentifiable due to the uncertainty of x. The possibility of being unable to identify a specific object except the index object is what it means by the possible unidentifiable. However, the set of natural numbers is an exception with this characteristic because its cardinality and ordinal constitute a bijection, such exception is caused by the apriority of counting numbers [1,5]. Therefore, any members of the natural number are always identifiable.

These three characteristics will constitute the main definition of metaphysical infinity. This will not deny it is a complex definition, but it is because when people attempt to use just a couple of sentences to define metaphysical infinity, there will be the unavoidable problem of self-referencing [6]. For instance, 'An infinity is a group of infinity objects'. Hence, the characteristics would be a better tool for clear understanding and effective usage in the following parts of my essay.

## 3. Set Theory Infinity

There are some mathematical notions for the understanding of set theory infinity. This chapter will introduce a clear explanation. Begin with the set theory, which is the mathematical theory concerning the notion of a Set (an object that consisting a group of objects), including the study of the set, members (or elements), and the relationship between members. Under the contemporary discussion about formulations of mathematics, most of the discussions are using set theory to describe various mathematical objects. Hence, the set theory has also been seen as one of the foundations of

mathematics. Around the notion of set, a series of mathematical notions were proposed. A subset, a subset is a partial set for a set, a set A is a subset of B if and only if, all members of A, are also members of B. Proper subset, a proper subset is a partial but not identical set of a set, a set A is a proper subset of B if and only if A is a subset of B while A is not identical to B. Powerset, a powerset $\varphi(A)$ is a set including all subsets of a set [6].

Concerning the numbers involving the notion of set, there are three notions. Firstly, the cardinal number, for any set, the amount of how many members in the set is the cardinal number of the set, and, for a finite set, its cardinal number would be a natural number. For instance, the cardinal number of a set $\{2,4,6\}$ is 3. In an infinite set (i.e. for the transfinite cardinal numbers), the cardinal number will be denoted as $\aleph$ followed by a subscript, describing the sizes of infinite sets. Secondly, the cardinality, the abstraction of the cardinal number is cardinality. It can be viewed as an abstract description of the number of members of a set. For any set, all the members of the such set would constitute a cardinality that can be analyzed via functional relationship. For instance, consider two sets A and B, when A and B are equinumerous ($A \sim B$, equal cardinality) if and only if there is a bijection $f: A \to B$ between A and B. i.e. For all members of A, there is a one-to-one correspondence via a function to all member of B. Thirdly, the ordinal, ordinal is a number defined concerning how a set is counted. Unlike the cardinal number which has been used to describe the size of a set. The ordinal number is used to describe the 'position 'of the members within a set (i.e., how a set arranges its members). An ordinal is defined starting with 0, labeling the first member that counted, and the following ordinal is defined by all ordinals that precede it. For instance, a set $\{x, y, z\}$ where the ordinal for x is 1, y is 2 and z is 3. The ordinal for a finite set is always a natural number, however, there can be a transfinite form like the cardinal number, usually will be denoted as $\omega$ and $\omega + n$, where the $\omega$ means the first ordinal that beyond natural numbers and n means the positions of the following ordinals [6].

For how the set theory defines infinity. In modern mathematics, the majority of mathematicians merely admit the existence of infinity and define it via set theory. The most widely accepted contemporary axiom system of set theory is the ZFC (Zermelo–Fraenkel and Axiom of Choice) set theory. Note that set theory only provides valid methods to construct sets and simply admits the validity of sets with infinite members, without specifically clarifying the nature of infinity. Take the set of natural numbers $\mathbb{N}$ as an example, ZFC allows a way to construct the set of natural numbers by, firstly admitting the existence of the empty set $\emptyset$ and using it as 0, then accessing any natural number n by recognizing it as the set $\{\ldots\{\emptyset\}\ldots\}$ with n braces, and finally stating that there exists a set of all these natural numbers. In this sense, most mathematicians avoid dealing with the existence of infinity and understand it in a metaphysical sense but merely use it with some well-defined entities such as the set of natural numbers.

However, mathematicians indeed tried to define the properties of infinity they encountered in mathematics without knowing its nature. A well-known example might be the concept of Dedekind Infinity. It says if a set is equinumerous to its proper subset, People.e., if each member of a set constitutes a bijection to each number of its proper subset, then the set is Dedekind Infinity. Intuitively, if a set has only finite members, then its cardinality must be larger than any of its proper subsets, so the concept of Dedekind Infinity must at least in some sense capture some nature of infinity.

In summary, the main distinction between metaphysical infinity and set theory infinity is the different foundation, the former is based on the very intuition and conceivability of mankind, while the latter is based on the axiom system that serves a functional purpose for mathematics [7].

## 4. Metaphysical Interpretation for Set Theory Infinity

In this part, It will argue that the axiom foundation of set theory infinity is not necessary in the sense that philosophers introduce set theory infinity to describe metaphysical infinity, there could be a proper metaphysical interpretation for set theory infinity, justifying it is identical to the metaphysical

infinity. To do so, It will explain how the three characteristics of metaphysical infinity can be interpreted from set theory infinity.

For periodic, it seems to be also an analytical characteristic for set theory infinity, which means when people grasp a set theory infinity, it must be periodic. For instance, the natural numbers, metaphysically, the 'means 'that link each member of natural numbers is the intuitive way of counting, when people denote something as 1, 2, and 3, etc., there must be a reason (mean) that they denote these things together, and in the case of natural numbers, what am People denoting is the numbers, People denote a number as 1, and another number as 2, etc. However, some might ask, how could simply denoting create some 'means', although People have the reason to denote 1, 2, and 3 because they are all numbers. But, there is no reason to distinguish one number from another one. So, what is the metaphysical reason that makes me distinct numbers? People think the reason is time, time is prior knowledge which means when People think, there must be time involved. Consider a circumstance, people think about Kant yesterday, and People think about Kant now. Thinking about Kant is the thing that they think of and it does not change due to time passage, however, the time passage provides me with a reason to distinguish think about Kant [1, 8], the first time People think about Kant and the second time they think about Kant. This is the same metaphysical reason why they could denote natural numbers, i.e., the intuitive way of counting. The axiom foundation that provides periodic for set theory infinity is the successor function, it states, for any natural number m and its successor natural number n, $m + S(n) = S(m + n)$. The axiom foundation does account for time passage, but it is metaphysically identical to the intuitive way of counting, hence, periodic is also an analytical characteristic for set theory infinity. At last, there might be some cases of set theory infinity that do not present clear periodic as the natural numbers, for instance, $\pi$ (irrational number). The 'means' for these cases is the way of calculation, for instance, the calculation method of $\pi$ is the means that link up every object.

For continuity. Consider the set of $\mathbb{R}$ Real Numbers, it has two line segments A:= (-2, 2) and B:= (-4, 4) as its subsets. A is the proper subset of B, $f(x) = 2x$ is the bijection between A and B. Because there is a bijection, A and B are equinumerous, i.e., B is a Dedekind Infinity. Noted that B in the metaphysical sense is a divisional infinity, which means there are infinite real numbers within the segment (-4, 4). Now, consider two finite scatter plot sets, $A_s := \{-2, -1, 0, 1, 2\}$ and $B_s := \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$. If people apply the function $f(x) = 2x$ to the finite sets $A_s$ and $B_s$, there is not an equinumerous between $A_s$ and $B_s$, i.e., $B_s$ would not be a Dedekind Infinity. So, why $f(x) = 2x$ make two subsets $A$ and $A_s$ produce different relationships with their main set $B$ and $B_s$, in which the former is equinumerous and the latter is not? When they use $f(x) = 2x$ to stretch a finite set, it will make its points sparse, but stretching an infinite set will not change the set's density on the line segment. The density of a Dedekind Infinity is unchangeable, and this can be interpreted by continuity because there are always new members, hence, stretching will not impact the density of the set on the line segment.

For unidentifiable. The set theory infinity does not claim any of its members is possible unidentifiable, while in some cases such as the set of real numbers, it will claim some of its members are uncountable but can be identified via a method such as Cantor's diagonal argument, hence the interpretation about the unidentifiable of set theory infinity is not as clear as previous two characteristics. Therefore, if people want to ask the reader to reconsider the previous two characteristics, typically, when confronting a finite set with periodic and partial continuity (the continuity before one counting to the set's end), there could be another metaphysical characteristic. And this characteristic indicates a partial structure of the set is identical to the entire structure of the set, which they want to term structurality. For instance, a set $\{2,4,6,8,\ldots,24\}$, if they take a subset that presents periodic and continuity, such as $\{2,4,6\}$, this subset indicates the structurality of 'adding 2 natural numbers to the former member will get the later member'. Hence, the partial structure is identical to the entire structure because the rest member of this set is still following such structurality [2].

Now, let it assume the structure still works with Dedekind Infinity. Suppose People have two sets, the even numbers, and odd numbers, if People use the structurality to compare the size of two sets, the even numbers will always be bigger than the odd numbers by their cardinality. Because the structurality of natural numbers indicates that for any odd number, there will be an even number following it [9]. However, Dedekind Infinity or set theory infinity explanation for the size of even numbers and odd numbers is they are equal. Because they are both proper subsets that are equinumerous to the natural numbers. Therefore, the equinumerous is the key part 1 establishes the metaphysical interpretation of unidentifiable. Consider why there is an equinumerous between the set of natural numbers and its proper subset, by definition, the proper subset is the subset that does not identical to the set, and equinumerous means for each member of set A, there is a bijection to each member of set B. For a finite set, its proper subset cannot achieve bijection to the set due to they are not identical by definition, but for a Dedekind Infinity, because there is not a finite (clear) boundary that limited the bijection, its proper subset can achieve bijection with the set [10]. Let me translate into the metaphysical language, the structurality only works under a known entire structure, and there is not a certain entire structure for an infinity. Therefore, one could not assert infinity possesses structurality. Back to the set theory Infinity, the equinumerous also indicates that set theory infinity does not have a known entire structure, hence the equinumerous can be achieved. The reason for an unknown entire structure for infinity is the possible unidentifiable because there are always objects except the index object is possible unidentifiable, hence, the entire structure of infinity lies on uncertainty, which cannot be known [10].

## 5. Conclusion

In conclusion, This article defined the metaphysical infinity with three characteristics, periodic, continuity, and unidentifiable. All infinity can be described by set theory, as the set theory infinity can be interpreted by metaphysical infinity via its three characteristics. Although the set theory infinity is based on the axiom foundation, it is also possible to use it to describe the infinity that is metaphysically conceivable and fits our intuition without axiom as the foundation. It is argued that the infinity described by set theory can be introduced into the metaphysical understanding of the nature of infinity.

## References

[1] Schechtman, A. Three Infinities in Early Modern Philosophy. Mind, 2019, 128(512): 1117–1147. DOI: https://doi.org/10.1093/mind/fzy034

[2] Soteropoulos, People. Metaphysics of infinity: The problem of motion and the infinite brain. UPA, 2013.

[3] Aristotle, A., & Aristotle. Metaphysics (Vol. 1, p. 9). Cambridge, MA: Harvard University Press, 1993.

[4] Usó-Doménech, J.-L., Selva, J. A. N., & Requena, M. B. Mathematical, Philosophical and Semantic Considerations on Infinity (People): General Concepts. Foundations of Science, 2015, 21(4): 615–630. DOI: https://doi.org/10.1007/s10699-015-9428-9

[5] Linnebo, Øystein, & Shapiro, S. Actual and Potential Infinity. Noûs (Bloomington, Indiana), 2019, 53(1): 160–191. DOI: https://doi.org/10.1111/nous.12208

[6] Bussotti, P., & Tapp, C. The influence of Spinoza's concept of infinity on Cantor's set theory. Studies in History and Philosophy of Science. Part A, 2009, 40(1): 25–35. DOI: https://doi.org/10.1016/j.shpsa.2008.12.010

[7] Pantsar, M. In search of ℵ₀: how infinity can be created. Synthese (Dordrecht), 2015, 192(8): 2489–2511. DOI: https://doi.org/10.1007/s11229-015-0775-4

[8] Kuhn, G. J. Metaphysics and mathematics: perspectives on reality. Hervormde Teologiese Studies, 2017, 73(3): 1–8. DOI: https://doi.org/10.4102/hts.v73i3.4663

[9] Krátká, M., Eisenmann, P., & Cihlář, J. Four conceptions of infinity. International Journal of Mathematical Education in Science and Technology, ahead-of-print(ahead-of-print), 2021, 1–25. DOI: https://doi.org/10.1080/0020739X.2021.1897894

[10] Usó-Doménech, J. L., Nescolarde-Selva, J., Belmonte-Requena, M., & Gash, H. Walking Through Cantor's Paradise and Escher's Garden: Epistemological Reflections on the Mathematical Infinite (People). Cybernetics and Systems, 2015, 46(6-7): 423–437. DOI: https://doi.org/10.1080/01969722.2015.1038473