The Reliability of Mathematical Reflection on the Thinking Structure of the World and Human Beings: Will Intuitionism Replace Logicism as the Foundation of Mathematics?

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Abstract. Logicism has prevailed with the development of modern technology innovation for decades. This is true, especially in the research field of Artificial Intelligence and Machine Learning. On the contrary, the development of another branch of mathematical thought, Intuitionism was not so optimistic. No matter how large the gap between Logicism and Intuitionism is in the current development of several research fields, the debates and challenges that Intuitionism raised to the domination of Logicism are worthy of fundamental scrutiny. This article will not adopt a simplistic view of the two schools but compare them through their arguments upon whether mathematics is reliable to reflect or to tell the thinking structure of the world and human beings. It is concluded that the debate over the authenticity of the mathematical foundation is far from drawing a clear and satisfactory answer.

Keywords: Logicism; Intuitionism; the Foundation of Mathematics.

1. Introduction

The 19th century was a crucial period in which many fields moved from classical to modern. Mathematics was one representative field during that period. The foundation for mathematics was built and shaken and has been shaky ever since. In the 1870s, to reduce classical mathematics to logic, Cantor created set theory, which would be considered to be the foundation for modern mathematics after development, and logicism was considered the main school of mathematics. In 1897, Forti revealed the first paradox in set theory and Cantor found a very similar paradox two years later. In 1902, Russell discovered the famous Russell's paradox and it shook the foundation for mathematics, brought the third mathematical crisis subsequently, and left mathematics without a firm foundation [1].

The three most important philosophies of mathematics, logicism, intuitionism, and formalism have made a great influence on many offshoots of mathematics and they all made an effort to give mathematics a solid foundation. Among these three schools, logicism and intuitionism had the biggest disagreement. Logicists thought that paradoxes in mathematics were caused by faulty mathematicians but not mathematics. On the other hand, intuitionists believed that those paradoxes indicated that classical mathematics was not a perfect mathematical system and that mathematics had to be rebuilt.

Many philosophers and mathematicians have made their perspectives about the foundation for mathematics in their works. Such as Principia Mathematica by Alfred Bertrand Russell and North Whitehead, The Foundations of Mathematics by Frank Ramsey, and The Continuum by Hermann Weyl. In this article, those perspectives and the development of logicism and intuitionism, and the differences between these two schools will be discussed to see which one is more likely to give the foundation for mathematics. Although some concepts of both schools are widely used nowadays, like set theory in computer science and the intuition of time in quantum physics, it is still very difficult to prove that everything in mathematics is a part of logic or from human beings' mental intuition systematically and completely. For most mathematicians, stating the nature of math as a transcendental system of truth is enough from a practical perspective.

2. Logicism

The discovery of logicism started with Leibniz in the 1700s. His calculus ratiocinator resembles symbolic logic and his way of solving problems was logical. The only way, in his opinion, to correct arguments was to make them as concrete as those of mathematicians, so errors could be spotted immediately, and people could simply calculate to discover who was right without more ado when there were disagreements between them [2]. After Leibniz, George Boole created The Mathematical Analysis of Logic which was considered to be the lighthouse of modern logic and mathematics. Frege attempted to develop a symbolic language to express mathematical propositions and carry out deductions in Concept Writing in 1879. And symbolic language was developed to be more convenient by Peano. In 1910, Principia Mathematica, which was written by Russell and Whitehead, showed that all classical mathematics could be deduced from set theory and it became the landmark of logicism. Finally, in 1922, Zermelo-Fraenkel set theory with the axiom of choice included (ZFC) was created and became the foundation of modern mathematics. The set theory was developed to be more complete in ZFC than it was in Principia Mathematica.

To reduce mathematics to logic is the basic idea of logicism. One of the primary goals of Principia Mathematica, according to Russell and Whitehead, was to demonstrate that all pure mathematics dealt only with ideas that could be expressed in terms of a very small number of basic logical concepts and that all of its propositions could be derived from a very small number of fundamental logical principles [3]. Stephen Korner said that the program of logicism was to derive mathematics from logical principles rather than from ideas that were partially logical and partially not [4]. To understand logicism, it is necessary to understand logic. All theorems that can be proved using first-order logic (FOL), which includes axioms, connectives, and deduction rules, are included in the definition of logic. FOL is a symbolic language or a formal language. It expresses pure logic and it is irrelevant to natural languages. Specific axioms and propositions are generated if natural languages are added to the foundation, and theorems are derived from those axioms and propositions. For example, if P is a proposition, then either p or not p is true according to the law of the excluded middle. In FOL the law is written as $P \vee \neg P$, where $\vee$ is the symbol of disjunction which means "or" and $\neg$ is the symbol of negation which means "not", and the value of $P \vee \neg P$ is true. Proposition P can become specific with natural languages. If P is set as "it is rainy today" in English, then the law of the excluded middle can be expressed as "it is true that either it is rainy today or it is not the case that it is rainy today". It is important to notice that this law or "the value of $P \vee \neg P$ is true" holds all the time for any proposition P in logicism. These kinds of laws or expressions in FOL are called logical propositions. A logical proposition is true by virtue of its form rather than its substance and is completely general. In other words, it does not matter what P is in natural languages in the example above, the law always holds with its form $P \vee \neg P$ and complete generality.

By applying logic, Cantor created the first set theory which is called a naive set theory. Although there were several paradoxes found in it (like Russell's paradox mentioned in the Introduction part), logicists believed that those paradoxes were not caused by classical mathematics itself but caused by erring mathematicians who used improper theories to explain classical mathematics. And the naive set theory was considered one improper theory. So, logicists after Cantor developed a more complete set theory which was the Zermelo-Fraenkel set theory with the axiom of choice included (ZFC). In ZFC, those paradoxes in naive set theory were all explained and fixed. And it seemed like everything in classical mathematics could and had been reduced to logic. In other words, at least on the surface, the idea of logicism was proven by logicists and logicism became the foundation for mathematics.

3. Intuitionism

This school was started in the 1900s by Brouwer. Unlike logicists who thought that classical mathematics was complete and just wanted to show that it is a part of logic, Brouwer thought that classical mathematics had some holes and it needed to be rebuilt. Stephen Cole Kleene created

recursive realizability in the late 1940s to understand intuitionistic claims and presented a more rational analysis of intuitionism in Introduction to Meta-mathematics.

In intuitionism, as opposed to the finding of underlying principles supposedly present in an objective reality, mathematics is thought to be only the result of the constructive mental activity of humans. To make it short, mathematics should be viewed as a mental activity rather than a collection of theories. Intuitionists believe that natural numbers like 123 are from the mental activity of humans about time, and all people possess a primal sense for the natural numbers that reside within them. After being immediately certain of what is the meaning of number one, the thought process that led to number one can be repeated in order to understand number two. Human beings can repeat this process to get the concept of all natural numbers. In intuitionism, this mental construction of natural numbers of human beings is believed to be inspired by time, since time is one-dimensional, the later time is after the early time. And it is impossible to have such mental construction of natural numbers if humans have no awareness of time. Brouwer must have been influenced by Immanuel Kant. Kant used the term "intuition" to refer to "instant consciousness," which is where the term "intuitionism" originated [1].

4. Similarity and Differences

The discovery of logicism started with Leibniz who is mentioned in part 2. In addition, to being a mathematician, Leibniz was also a great philosopher in metaphysics. As we see it now, his ideas are not only about logicism but also relevant to intuitionism. He considered that if people could discover characters or signs that could express all of the human's thoughts as precisely as arithmetic expressed numbers or geometry expressed lines, people would be able to accomplish in all areas of reasoning everything that arithmetic and geometry could accomplish [5]. He thought the symbolic logic of mathematics was not enough to describe the world. If more basic characters, which represented human beings' thoughts or mental activity, were created, everything, which includes mathematics, can be expressed and explained. The perspective from Leibniz shows that, in a sense, mathematics is a combination of logic, which refers to logicism now, and mental activity, which refers to intuitionism now. For some scholars like Frank Ramsey, the boundary between logicism and intuitionism is clear to them. They stick on their sides. Ramsey believed that mathematics was a part of logic, so it belonged to logicism as opposed to the formalist and intuitionist schools [6]. However, for some other scholars, it is just the opposite. For example, Hermann Weyl gave up on logicism and joined Brouwer in the intuitionist movement.

Some principles of logic that are not made up of constructs, in the opinion of intuitionists, are meaningless. The law of the excluded middle which is mentioned in part 2 is one example. In logicism, the form of the law, $P \vee \neg P$, is always true, which means that one side of the disjunction can be proven to be true by showing the fault of the other side. This is also called reductio. However, in intuitionism, proof must be constructed for the side which is needed to be proven. It makes no sense to say "case P" is true by showing that "not case P" is not true. Intuitionists negate the equivalence between "case P" and "not case P", so the law is meaningless to them.

To be more specific, here is a proof question. "Is there a pair of irrational numbers a and b, such that ab is rational?" Logicists will probably prove it in this way:

Proof 1

$$\text{let } a = e,\ b = \ln 2, \text{ then } a^b = e^{\ln 2} = 2$$

$$e \text{ is a transcendental number that is irrational}$$

$$\text{want to show that } \ln 2 \text{ is irrational}$$

$$\text{suppose that } \ln 2 \text{ is rational, then } \ln 2 = \frac{m}{n}$$

$$(m, n \in Z, n \neq 0, m,n \text{ have no common factor})$$

$$\text{then } 2 = e^{\frac{m}{n}} \Rightarrow 2^n = e^m$$

$$2^n \text{ is an integer} \Rightarrow e^m \text{ is an integer, which is a contradiction}$$

So $\ln 2$ is irrational (the law of excluded middle), so there is a pair of irrational numbers a and b, such that ab is rational.

The proof is done.

Or in this way:

Proof 2

$$\text{let a} = \sqrt{2}\ , \text{b} = \sqrt{2}\ (\sqrt{2} \text{ is irrational})$$

$$\text{case 1: if } \sqrt{2}^{\sqrt{2}} \text{ is rational, then the proof is done}$$

$$\text{case 2: if } \sqrt{2}^{\sqrt{2}} \text{ is irrational, then let a} = \sqrt{2}^{\sqrt{2}}, \text{b} = \sqrt{2}$$

$$\text{then a}^{\text{b}} = \sqrt{2}^{\sqrt{2}^{\sqrt{2}}} = \sqrt{2}^{\sqrt{2}\times\sqrt{2}} = \sqrt{2}^{2} = 2$$

Since 2 is rational, the proof is done, either case 1 or case 2 is true, the proof is done

Intuitionists will never agree with the two proofs above. Firstly, the law of excluded middle is used as proof 1. Why $\ln 2$ is rational is not proven directly. Secondly, proof 2 is non-constructive, which means that it proves the existence of a particular kind of object without providing an example. These proofs could be a little magical, but they can also make people feel tricked. Just like proof 2, it does not give a way of telling which pair of numbers has the required property [7]. On the opposite side, as mentioned in Part 3, intuitionists believe that mathematics is from the constructive mental activity of humans, so all proofs in mathematics must be proven in a constructive way that demonstrates the existence of an object by creating or providing a method for creating the object.

Two schools also disagree on infinity. In set theory, which belongs to logicism, there are different sizes among infinite sets. For instance, there is not a one-to-one correlation between each real number and each natural number, the set of all real numbers R is greater than the set of all natural numbers N. In logic, an infinite set is said to be "countable" if it can be arranged in one-to-one correspondence with the natural numbers; otherwise, it is "uncountable". However, in Introduction to Metamathematics, Kleene mentioned some ideas from Brouwer. According to Brouwer, there is no evidence to back up the notion that the totality of all natural numbers has an existential quality. The sequence of numbers is a universe of possibilities that is open to infinity and grows beyond any level already attained by moving on to the next number. The sequence is perpetually in the state of creation but not a closed universe of entities that exist by themselves [8]. This closed realm refers to the set of natural numbers. Brouwer disagrees with logicists on whether it is reasonable to consider all natural numbers as a complete set with certain properties like their size of it.

5. Disadvantage of Logicism

Logicism is not firmly supported by mathematicians even for some mathematicians were once logicists. For example, Hermann Weyl thought that the principle of iteration could not be reduced to purely logical principles. Since if it could, then it would not be a specifically mathematical principle [9]. And some mathematicians consider there are some problems with the theories of logicists. Godel showed the formal system of Principia Mathematica was doomed to failure from the out sets. It was recognized that although indeed parts of mathematics can be formulated in logical terms, there was something that separated mathematics from logic. Some mathematical principles could not be adequately captured by logic alone. This kind of failure is not just in Principia Mathematica. Even in ZFC, logicists did not reduce all of mathematics to elementary logic, but they did do so by adding the theory of properties (or sets), properties of properties, and so forth to elementary logic. Thus, math can be reduced to logic if property theory (or set theory) is considered as an element of logic [10].

To reduce mathematics to logic is the basic idea of logicism, but intuitionists think that mathematics cannot be reduced to any other things like logic. This is a significant gap between logicism and intuitionism. Logicists want to show that mathematics is a part of logic. On the intuitionist's side, they merely believe that any mathematics that cannot be solved intuitively is not mathematics. If there is something in mathematics which is needed to be proven, for intuitionists, if

they can construct that thing, then it is in mathematics. If they cannot construct it, then it is just meaningless and they will not be concerned about it anymore. However, for logicists, if they can prove that thing is true, then it is in logic. If they cannot prove it, they need to think out a way to prove it, then it is in logic. So, compare to intuitionists, logicists have a more challenging task. They need to prove that mathematics is a part of logic, but intuitionists can just wait and see what comes out of their definition of mathematics.

6. Conclusions

There is no conclusion about which of the schools is the foundation for mathematics yet. Scholars from each of the schools hold their position and sometimes they change the side. Logicists need to prove that intuitionism is wrong or it is a part of logic to show that logicism is the foundation for mathematics, and they are more likely to do the former because they believe that intuitionism stands on the opposite side of logicism. Logicists had made some points to reject intuitionism like if something can be proven in both schools, the proof in intuitionism is always much longer than it is in logicism, but those kinds of points are neither rational nor scientific. Intuitionists, can just sit back and wait for the source of their idea, which is time, to prove everything. Additionally, according to the situation now, mathematically minded philosophers and philosophically minded mathematicians have moved on. They lose interest in arguing the foundation for mathematics but focus on mathematics other than its foundation. It is comparable to what John Von Neumann once told Felix Smith: "Young man, in mathematics you don't understand things. You just get used to them".

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