A study of risk free and risky portfolio investment optimisation strategies under multi-objective constraints

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Abstract. The purpose of portfolio investment is to maximise returns while minimizing risks. For investors with different risk attitudes, this research uses Economic Model Predictive Control (EMPC) to study the multi-objective securities portfolio problem and obtains a multi-objective securities strategy that guarantees the maximum return and the minimum risk under different risk attitudes through the tracking method, considering the return, risk, and other factors of the securities portfolio. Finally, the effectiveness of the strategy is verified by simulation, and an analytical method for selecting the optimal portfolio of securities is proposed.

Keywords: Security investment; portfolio optimization; investment risk; investment return; investment strategy.

1. Introduction

The securities investment portfolio is to invest the holding funds in several assets, such as stocks and bonds, to achieve the investment goal of maximizing returns and minimizing risks, Return and risk are factors that affect investors' investment decisions, and the investment objective of minimizing risk is considered. As investors' different attitudes towards risk affect their trade-off between return and risk, the investment decision to maximise return and minimise risk simultaneously for investors with different risk attitudes is investigated by introducing a risk aversion coefficient [1]. Optimal investment strategies for multi-stage portfolio optimisation problems are proposed using the control theory, where model predictive control is used in a rolling optimisation approach and has a high capacity to deal with constraints; and use economic model predictive control to formulate securities portfolio strategy to achieve investment goals [2].

The tracking method is used to deal with multiple conflicting objectives in model predictive control. The computational process of this method does not depend on the construction of the steady state pareto frontier and is easily applied to practical portfolio systems [3]. However, in solving multi-objective of maximizing returns and minimizing risks conflict with each other, the investment objective function is not zero at the investment equilibrium point [4]. The optimal portfolio strategy obtained by model predictive control is usually fails to achieve stable portfolio wealth. Therefore, the multi-objective portfolio problem can be solved as a special kind of model predictive control problems [5].

The concept of "Economic Model Predictive Control (EMPC)" demonstrates that when the cost function has a non-zero gradient at the optimal steady state, the model predictive control satisfied the regularity condition, the economically optimal nominal stability cannot be achieved [6]. Therefore, the average performance of EMPC must be worse than that of tracking model predictive control [7]. To address this problem, his consistent exponential stability of the economically optimal steady state is improved by introducing a linear terminal penalty to correct the steady state point gradient [8]. The positive definite auxiliary function of the optimal equilibrium point is used to obtain a stable multi-objective economic model predictive control strategy with multiple economic performance indicators using the tracking method, for a class of non-linear switching systems with unreachable states [9]. A dissipation based EMPC auxiliary economic indicator is used to obtain a switching control strategy that approximates the original economic performance indicators. For constrained non-linear systems with bounded perturbations, the differential game theory is used to optimize the conflicting economic performance index and robust and stable performance index to ensure the closed-loop system under bounded disturbance stability [10].

In this paper, a portfolio strategy is proposed based on the EMPC approach. By constructing a cost function with terminal penalty and solving a finite-time domain EMPC problem with terminal constraints, a multi-objective portfolio strategy that satisfies both return maximisation and risk minimisation under different risk attitudes is obtained.

2. Portfolio model

Considering a portfolio consisting of 'n' risky assets and '1' risk-free asset. Assume that the investment process consists of several stages, at each of which the investor acquires information about new assets and responds to the new market state by buying or selling some of them (short selling is allowed) [11].

The buy amount u+i(k) and the sell amount u-i(k) of the i^{th} risky asset at time k in the model are combined and denoted as $u_i(k)$, where $u_i(k)>0$ and $u_i(k)< 0$ are the amounts of the i^{th} risky asset bought and sold by the investor at time k respectively, and $u_i(k) =0$ indicates the case of no trade. Then the value of the investor's holding of the i^{th} risky asset is

$$x_i(k+1) = (1+\eta_i(k+1))(x_i(k)+u_i(k)), i = 1,2,\cdots,n, \tag{1}$$

Where $\eta_i(k+1)$ is the rate of return on the i^{th} risky asset at [k, k+1]. Similarly, the value of a risk-free asset held by an investor is

$$x_{n+1}(k+1) = (1+r_1)(x_{n+1}(k)+u_{n+1}(k)) \tag{2}$$

Where r_1 is the rate of return on the risk-free asset and $u_{n+1}(k) \geq 0$ is the amount invested in the risk-free asset at time k.

Assuming that the investor borrows funds when needed, the dynamics of the amount borrowed is

$$x_{n+2}(k+1) = (1+r_2)(x_{n+2}(k)+u_{n+2}(k)) \tag{3}$$

Where: r_2 is the risk-free borrowing rate and $r_1<r_2$; $u_{n+2}(k)$ is the amount borrowed at time k.

In summary, the total portfolio wealth at time k is expressed as

$$y(k) = \sum_{i=1}^{n+1} x_i(k) - x_{n+2}(k) \tag{4}$$

For the portfolio system formula (1)-(4), let $x(k)=[x_1(k)x_2(k)\cdots x_{n+2}(k)]^T$ be the state vector of the portfolio system at time k, $u(k)=[u_1(k)u_2(k)\cdots u_{n+2}(k)]^T$ be the input vector, and the total portfolio wealth y(k) be the output variable at time k. The portfolio model can then be written in the following state space form:

$$\begin{cases} x(k+1) = A(k)x(k) + B(k)x(k) \\ y(k) = Cx(k) \end{cases} \tag{5}$$

Where $A(k)$=diag, $\{1+\eta_1(k),\ldots,1+\eta_k(k),1+r_1,1+r_2\}$ is the coefficient matrix; B(k) is the input matrix satisfying $B(k) = A(k)$; $C = [1 \ldots 1 -1]$ is the output matrix.

Since the value of the risk-free asset and the amount borrowed are at least zero (if the value of the risk-free asset is less than zero, investors will not invest; if the amount borrowed is less than zero, this asset becomes a risk-free asset with return r_2), then $x_{n+1}(k+1) \geq 0$, $x_{n+2}(k+1) \geq 0$. Since short selling is allowed, let the amount of short selling be $d_i(k) \geq 0$; to ensure the order of securities market trading, assume that the transaction amount is restricted. Thus, it is obtained that:

$$\begin{cases} x_{n+1}(k) + u_{n+1}(k) \geq 0, \\ x_{n+2}(k) + u_{n+2}(k) \geq 0 \\ x_i(k) + u_i(k) \geq -d_i(k), i = 1,2,\ldots,n, \\ u_{min} \leq u(k) \leq u_{min}, \end{cases} \tag{6}$$

Where u_{min}= [$u_{min,1}u_{min,2}\cdots u_{min, n+2}$] ^{T}, u_{max}=[$u_{max,1}$ $u_{max,2}$ … $u_{max,n+2}$]^{T}, $u_{min,i}$, $u_{max,i}$(i=1,2,⋯,n+2) are the minimum and maximum transaction amounts for each asset respectively.

3. Multi-objective portfolio problem

The investor's investment objective is to obtain a high investment return while keeping the investment risk as low as possible. Therefore, this paper investigates how to develop a portfolio investment strategy that enables the investor to maximise investment return while minimising investment risk in order to achieve the desired investment objective.

3.1. Portfolio problems considering return maximisation and risk minimisation.

Investment objective of maximising returns:

$$\min_{\boldsymbol{u}(k)} L_1(\boldsymbol{x}(k), \boldsymbol{u}(k)) = \min_{\boldsymbol{u}(k)} \left\{-\boldsymbol{\eta}^{\mathrm{T}}(k+1)(\boldsymbol{x}(k)+\boldsymbol{u}(k))\right\} \tag{7}$$

Where$\eta(k+1) = [\eta_1(k+1) \ldots \eta_n(k+1)\; r_1 - r_2]^T$.

By weighting the errors and transaction amounts between the actual portfolio wealth and the reference portfolio wealth to measure the risk of a portfolio of securities, the risk minimising investment objective is

$$\min_{\boldsymbol{u}(k)} L_2(\boldsymbol{x}(k), \boldsymbol{u}(k)) = \min_{\boldsymbol{u}(k)} \gamma(y(k) - V^0(k))^2 + \boldsymbol{u}^{\mathrm{T}}(k)\boldsymbol{R}\boldsymbol{u}(k) \tag{8}$$

Where: γ is the error weighting factor between the actual portfolio wealth and the reference portfolio wealth, R is the transaction amount weighting matrix; $V^0(k) = (1+\mu_0)^k\, V^0(0)$ denotes the reference portfolio wealth, $V_0(0)$ denotes the initial reference portfolio wealth and μ_0 denotes the reference rate of return.

Assume that at the initial moment of investment, the portfolio value is

$$x(0) = x_0 = \left[x_{0,1}\; x_{0,2 \ldots x_{0,n+2}}\right]^T \tag{9}$$

Where $x_{0,\,i}$ (i=1, 2…, n+2) is the initial value of each asset.

In the actual portfolio process, high returns are often accompanied by high risks, so the two investment objectives of maximising returns and minimising risks conflict with each other and often cannot be achieved simultaneously, therefore, at the end of the investment period the portfolio value can only be guaranteed to have the highest possible return and the lowest possible risk, i.e:

$$x(N) \in \;\; \Omega, \Omega = \mathrm{B}(x^s, \delta) \tag{10}$$

Where: N is the number of investment stages, x(N) denotes the terminal portfolio value; Ω denotes the range of the terminal portfolio value of the investment, and B (xs, δ) is the spherical domain with centre x_s (x_s is the steady-state equilibrium point) and radius δ.

Let $Z_{N,p} = \{(x(k), u(k)) |\, (x(k), u(k)) s.t. equation\ (6), \forall k \in Z_{0:N-1}, (x(N), u(N)) \in \Omega\}$, then$X_{N,p} = \{x(k) | exist\ u(k), makes\ x(k), u(k)) \in\ Z_{N,p}\}$.

The investment objectives are (a) to maximise return and (b) to minimise risk. As a result, the multi-objective portfolio problem is obtained as

3.2. Multi-objective portfolio problem of securities based on the tracking method.

In the process of securities investment, the investment objectives (a) and (b) are in conflict. For such predictive control problems involving two or more objectives, the tracking method is proposed to solve them.

Definition 1 : A feasible point (x^p,u^p) of a multi-objective problem is Pareto optimal when and only when there exists no other feasible point $(x(k),u(k))$,such that $L_i(x(k),u(k))\leq L_i(x^p,u^p),\forall i\in\{1,2\}$, and for at least one indicator $i\in\{1,2\}$, there is $L_i(x(k),u(k))< L_i(x^p,u^p)$

Definition 2: If the set of solutions of Pareto is not empty set, the boundary of the set in the cost space is called the Pareto front.

The steady-state Utopia point is determined by the solution $(x_i^{L,s}, u_i^{L,s})$ of the component $L_i(x_i^{L,s}, u_i^{L,s})$ (i=1,2) in the cost space, where $L_i(x_i^{L,s}, u_i^{L,s})$ (i=1,2) is given by the solution of problem (10). The Utopia point is denoted $L^{L,s}= [L_1^{L,s} \quad L_2^{L,s}]^T$, although the problem (10) the investment objective is conflicting, for instance, the Utopia point is unreachable, but the Utopia point can be utilised as a reference point.

The steady-state compromise solution (x^s,u^s) is given by the solution L $(x^s,u^s)$of the following minimum distance problem:

$$\begin{cases} \min\limits_{u(k)} \| \mathrm{L}(\mathrm{x}(\mathrm{k}),\mathrm{u}(\mathrm{k}) - \mathrm{L}^{\mathrm{L},s} \|^2_{Q(y(k),k)}, \\ \mathrm{s.t.} \begin{cases} \mathrm{x}(\mathrm{k}+1) = \mathrm{A}(\mathrm{k})\mathrm{x}(\mathrm{k}) + \mathrm{B}(\mathrm{k})\mathrm{u}(\mathrm{k}), \mathrm{y}(\mathrm{k}) = \mathrm{C}\mathrm{x}(\mathrm{k}), \\ \mathrm{x}(\mathrm{k}),\mathrm{u}(\mathrm{k})) \in \mathrm{Z}_{\mathrm{N,p}}, \mathrm{x}(0) = \mathrm{x}_0, \mathrm{x}(\mathrm{N}) \in \Omega, \end{cases} \end{cases} \tag{11}$$

Where:

$$\mathrm{L}\big(\mathrm{x}(\mathrm{k}),\mathrm{u}(\mathrm{k})\big)=\big[\mathrm{L}_1\big(\mathrm{x}(\mathrm{k}),\mathrm{u}(\mathrm{k})\big)\ \ \mathrm{L}_2\big(\mathrm{x}(\mathrm{k}),\mathrm{u}(\mathrm{k})\big)\big]^{\mathrm{T}} \tag{12}$$

$$\left\|\mathrm{L}\big(\mathrm{x}(\mathrm{k}),\mathrm{u}(\mathrm{k})\big)\text{-}\mathrm{L}^{\mathrm{L},\,\mathrm{s}}\right\|^{2}_{\mathrm{Q}(\mathrm{y}(\mathrm{k}),\,\mathrm{k})}=\big[\mathrm{L}\big(\mathrm{x}(\mathrm{k}),\mathrm{u}(\mathrm{k})\big)\text{-}\mathrm{L}^{\mathrm{L},\,\mathrm{s}}\big]^{\mathrm{T}}\ \mathrm{Q}(\mathrm{y}(\mathrm{k}),\mathrm{k})\ \big[\mathrm{L}\big(\mathrm{x}(\mathrm{k}),\mathrm{u}(\mathrm{k})\big)\big] \tag{13}$$

$\mathrm{Q}(\mathrm{y}(\mathrm{k}),\mathrm{k})=\begin{bmatrix} \mathrm{q}_1 & 0 \\ 0 & \mathrm{q}_2(\mathrm{y}(\mathrm{k}),\mathrm{k}) \end{bmatrix}$ Is the tracking weight matrix; $\mathrm{q}_2(\mathrm{y}(\mathrm{k}),\mathrm{k})=\frac{\mathrm{c}}{\mathrm{y}(\mathrm{k})}$ is the risk aversion factor, which measures the investor's attitude towards investment risk. In this way, the multi-objective portfolio problem (10) is transformed into a tracking problem shaped as in equation (11).

4. EMPC-based optimal portfolio strategy for securities

Since the investment objective (a) conflicts with (b), the objective function in problem (11) is not zero at the steady-state equilibrium point x_s, i.e. this objective function is not positive definite at the steady-state equilibrium point x_s. Therefore, the optimal portfolio strategy is formulated below using the multi-objective optimization based EMPC.

Let the economic performance function be

$$\mathrm{L_e}\big(\mathrm{x}(\mathrm{k}),\mathrm{u}(\mathrm{k})\big) = \big\|\mathrm{L}\big(\mathrm{x}(\mathrm{k}),\mathrm{u}(\mathrm{k})\big) - \mathrm{L}^{\mathrm{L},s}\big\|^2_{\mathrm{Q(yk),k)}} \tag{14}$$

To ensure that the terminal value of each investment asset is within the spherical domain Ω, a cost function is set for situations beyond this spherical domain to penalise portfolios away from the Utopia point. Further define the objective function as

$$V_N(x,u) = \sum_{j=0}^{N-1} L_e(x(k+j|k), u(\,(k+j|k)) + V_f(x(N|k)), \tag{15}$$

Where $V_f(x(N|k))$ is the terminal cost function that penalises portfolios far from the Utopia point. The multi-objective portfolio problem can then be expressed as an EMPC problem as follows:

$$\begin{cases} \min\limits_{u(k)} \mathrm{V_N}\ \big(\mathrm{x}(\mathrm{k}),\mathrm{u}(\mathrm{k})\big), \\ \mathrm{s.t.} \begin{cases} \mathrm{x}(\mathrm{k}+1) = \mathrm{A}(\mathrm{k})\mathrm{x}(\mathrm{k}) + \mathrm{B}(\mathrm{k})\mathrm{u}(\mathrm{k}), \mathrm{y}(\mathrm{k}) = \mathrm{C}\mathrm{x}(\mathrm{k}), \\ \mathrm{x}(\mathrm{k}),\mathrm{u}(\mathrm{k})) \in \mathrm{Z}_{\mathrm{N,p}}, \mathrm{x}(0) = \mathrm{x}_0, \mathrm{x}(\mathrm{N}) \in \Omega, \end{cases} \end{cases} \tag{16}$$

By solving the EMPC equation (17) at each sampling moment, an optimal control sequence is obtained:

$$u^*(k) = [u^*(k|k) \;\; u^*(k+1|k) \ldots u^*(k+N-1|k)]^T \qquad (17)$$

The first control $u^*(k|k)$ of this sequence is applied to the portfolio system, and the feedback control law is obtained as, $k(x(k)) = u^*(k|k)$, at which point the portfolio system (18) is.

$$\begin{cases} x(k+1) = A(k)x(k) + B(k)k\big(x(k)\big), \\ y(k) = Cx(k) \end{cases} \qquad (18)$$

5. Simulation experiments

In this section, an example is used to verify the feasibility of the control strategy. The simulation data are obtained from the New York Stock Exchange (NYSE) asset data for September 2016 to November 2019.

Consider an investor who invests in two risky assets and one risk-free asset and can borrow funds when needed. Assume that the interest rate on the risk-free asset is $r_1 = 0.002$, the interest rate on borrowing is $r_2 = 0.003$, and the forecast and control time horizons are 5.

At the start of the investment, it is assumed that the investor initially invests all the funds in the risk-free assets, so that the initial value of each asset is $x_1(0) = 0$, $x_2(0) = 0$, $x_3(0) = 1$, $x_4(0) = 0$, and the investor's initial wealth is $y(0) = 0$.

Assume that the amount of borrowed funds does not exceed y(k) and that short selling is prohibited, i.e. $d_0(k)=y(k)$, $d_1(k)=d_2(k)=0$, $u_{min}=[-1\ -1\ -1\ -1]^T$, $u_{max}=[1\ 1\ 1\ 1]^T$; take the actual the error weight coefficient $\gamma=0.2$ between the actual portfolio wealth and the reference portfolio wealth, the transaction amount weight matrix $R=\mathrm{diag}\ \{0.1,0.1,0.1,0.1\}$, the initial reference portfolio wealth $V_0(0)=1$, the reference return $\mu_0=0.001$; the Utopia tracking weight matrix for risk lovers, risk neutrals, and risk averse individuals respectively is $Q_1(y(k),k)=\mathrm{diag}\ \{0.8,0.2y^{-1}(k)\}$, $Q_2(y(k),k)=\mathrm{diag}\ \{0.5,0.5y^{-1}(k)\}$, $Q_3(y(k),k)=\mathrm{diag}\ \{0.2,0.8y^{-1}(k)\}$. The simulation results are shown in Figure 1 and Figure 2.

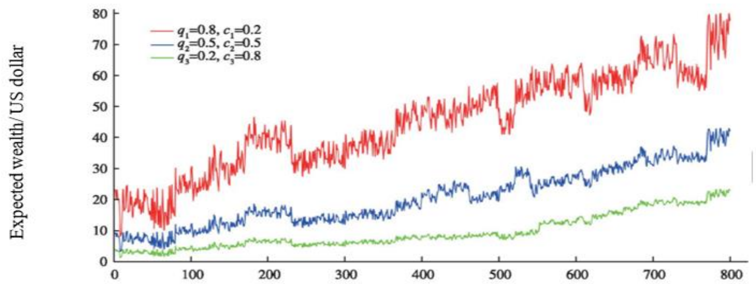

Fig 1. The expected return wealth of securities portfolio.

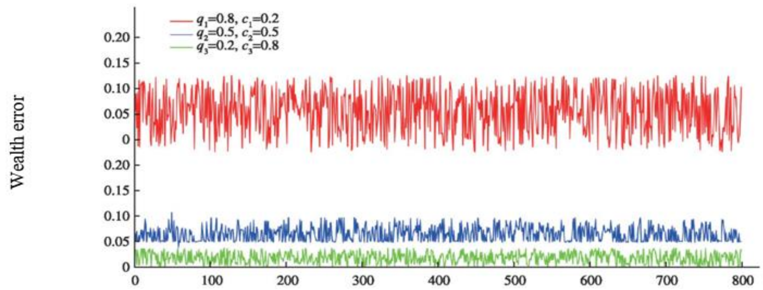

Fig 2. The square of the error between actual portfolio wealth and reference portfolio wealth.

Figure 1 shows that the overall portfolio wealth is on an increasing trend during the investment process, the designed portfolio strategy has positive returns, and the smaller the risk aversion factor, the higher the returns. Figure 2 shows that under the formulated investment strategy, the risk of portfolio wealth is at a low level and decreases as the risk aversion factor increases. Combining Figures 1 and 2, the portfolio strategy designed in this paper is a good balance between the investment objectives of maximising returns and minimising risks under different risk attitudes.

6. Conclusion

In this paper, an optimal portfolio strategy based on EMPC is proposed to solve the multi-objective portfolio control problem through the tracking method. Considering that there is a certain time delay when investors make investment decisions, the future mainly considers the securities portfolio problem with time delay.

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