Enhancing Option Pricing Accuracy through Empirical Performance: A GARCH-Based Approach to Refining Volatility Assumptions in the Black-Scholes Model

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Abstract. This study presents a comprehensive investigation into enhancing the precision of option pricing by refining the volatility assumption of the Black-Scholes (BS) model through the incorporation of the GARCH model. This study uses a methodical approach that includes rigors comparative analysis, Monte Carlo simulations, and historical data calibration. The findings indicate that, when the GARCH model is combined with the conventional BS model using implied volatility, the RMSE is reduced by about 3.3%. The slight autocorrelation in the sample data's inherent constraints, however, mitigate this improvement. This study emphasizes the significance of model assumption revision for capturing changing market dynamics, making a significant contribution to the conversation on improving option pricing accuracy. This study also acts as a crucial resource for academics and industry professionals navigating the complexity of option pricing and volatility modelling.

Keywords: Monte Carlo simulation, BS Model, GARCH model, autocorrelation.

1. Introduction

Financial derivatives, sophisticated products that have developed from crude risk management tools to flexible financial instruments, have had a significant impact on the field of modern finance. Options stand out among these derivatives due to their dual nature of rights, giving investors the freedom to buy or sell underlying assets at defined prices within predetermined timeframes. This trait gives options the special capacity to function as efficient tools for mitigating risk exposure and hedging against market volatility. Such risk mitigation techniques have been widely used, giving solutions with significant value across several financial sectors.

The price of options is essential to their effectiveness since it helps traders and risk managers create well-informed trading strategies and choices. The quantification of possible gains and losses, which informs investment choices and hedging tactics, depends critically on accurate option pricing. Financial institutions place great importance on this procedure since it helps them manage their portfolios more effectively and handle challenging market situations. The Black-Scholes (BS) model and the binomial tree technique are two widely used methods in option pricing. Because it provides closed-form answers for European-style options, the BS model, which is based on stochastic calculus, is a cornerstone of option pricing. But given that it is predicated on ideas like perpetual volatility and frictionless markets, there have been ongoing debates about how realistically accurate it is.

This research aims to delve into the accuracy of the BS model in option pricing. The primary approach will center on the utilization of Monte Carlo simulations. Initially, within the framework of the Black-Scholes model, the project will simulate the future path of Carnival Corporation's stock over a designated period. Based on the BS model, the prices of September call options for this stock at various strike prices will be computed. Subsequently, a GARCH model pertaining to volatility will be calibrated using historical data. This calibrated GARCH model will then be applied within the Black-Scholes framework to simulate the future trajectory of the aforementioned stock over the specified period. This will involve computing the corresponding prices of September call options at varying strike prices. Ultimately, a comparative analysis will be undertaken by contrasting the two sets of computed option prices with market prices. The evaluation of the improvements resulting from the application of the GARCH model will be determined using the Root Mean Squared Error (RMSE) metric.

For financial markets and institutions alike, improving option pricing accuracy has significant significance. Accurate option valuation helps investors make wise decisions and ensures they are fairly compensated for the risks they assume. Accurate pricing supports risk management tactics and protects portfolios from unforeseen market changes. Furthermore, sophisticated pricing techniques increase market effectiveness by promoting openness and trust among market players. Improved precision enables financial institutions to better manage capital and increase their risk-adjusted returns by more closely aligning valuations with market reality. In the end, a more precise option pricing structure improves the overall robustness and effectiveness of financial markets while enhancing market players' capacity for strategic decision-making.

2. Literature Review

2.1. Research on BSM Model

The option pricing model for non-dividend payments was first proposed by Black and Scholes. Starting from a set of partial differential equations, they derived the now famous Black-Scholes formula. The model, which makes the assumption that stock price movements occur according to a geometric Wiener process [1], expresses the relationship between option price, underlying asset price, volatility, time to maturity, and risk-free interest rate neatly. Merton expanded on the mathematical underpinnings of the Black-Scholes model, which advanced the idea of continuous-time modelling of financial markets, to complement Black-Scholes' work [2]. The volatility of the underlying stock, however, is a parameter in the BS model that cannot be immediately monitored in the market. In response to this realization, Black conducted a thorough analysis of the dynamic nature of stock price volatility and its effects on option pricing [3]. The BS option pricing model has undergone major advancements and expansions since its creation. One of the effective improvements is targeting volatility assumptions [4]. In a framework for option pricing, Lahouel and Hellara model the dynamics of asset returns using the modified GARCH process. The findings demonstrate that the affine GARCH model's outputs are consistently the optimum hedging strategy for optional portfolio hedging [4].

2.2. Research on GARCH Model

The traditional BS model makes the assumption that the stock's log return has a Gaussian distribution. However, one of the earliest works show that neither price variations nor log returns were Gaussian [5]. In order to account for empirical fat tails in log returns, Mandelbrot proposed a new i. i. d. distribution [5]. Modern financial experts and practitioners do, however, contend that the volatility of asset returns evolves with time. 'Fat' tails may result from this as well. Engle developed the Auto-Regressive Conditional Heteroskedastic process to address this issue [6]. The ARCH model is a straightforward dynamic volatility model, but to correctly represent the volatility process, it frequently needs a lot of parameters (lags)[6]. Thus, Bollerslev introduced the Generalized Auto-Regressive Conditionally Heteroscedastic (or GARCH) process to handle the above problem of fitting parameters [7]. He claims that the data appear to fit the model's most basic version—in which the conditional variance just depends on its own lag and the lagged squared innovation—the best [8]. Since then, a number of theoretical econometric research have been conducted in an effort to clarify the theoretical characteristics of GARCH models. Statistics and probability have made improvements to the the mathematical framework regarding the division of the jump and diffusive elements of the martingale in recent years [9]. Returning to our area of option pricing, Dritsakis & Savvas thought that the BHHH algorithm of the maximum likelihood approach can be used to estimate the parameters of the GARCH model in order to predict volatile stocks [10]. At the same time, they discovered that the ARMA (0,3)-GARCH-M (1,1) model best captured the sample volatility through empirical data analysis of the Nordic Stock Exchange [10]. According to Hansen & Lunde, GARCH (1,1) is the most effective and widely applied iteration of GARCH to date [11]. This is the main justification for using GARCH (1,1) for simulation in this paper.

3. Data and Method

3.1. Data

The three-year Carnival data from July 16, 2020 to July 16, 2023 was used in this article. The stock price for the following 43 trading days starting on July 16, 2023 is the goal of the Monte Carlo simulation. The option whose exercise date is September 15, 2023 corresponds to the simulated stock price for Carnival. The following fundamental data (table 1) pertains to Carnival's daily rate of return during the previous three years:

Table 1. Information of Carnival's Daily return

	Mean	std	Min	25%	75%	Max
Daily return	0.1113%	4.215%	-20.93%	-2.216%	2.376%	38.00%

The logarithm of Carnival's historical return rate is used in this article to examine the distribution of the log-normal return assumption made by the BS model. The figure 1 displays the distribution of the log return. The skewness of this distribution is 0.73, which is close to 0, indicating that the distribution of the data is roughly symmetrical and there is no obvious long tail. The kurtosis of this distribution is 9.78, which is significantly higher than the kurtosis of a completely normal distribution, which is 3. This demonstrates that the data is concentrated in a limited standard deviation range, and the kurtosis is high and acut.

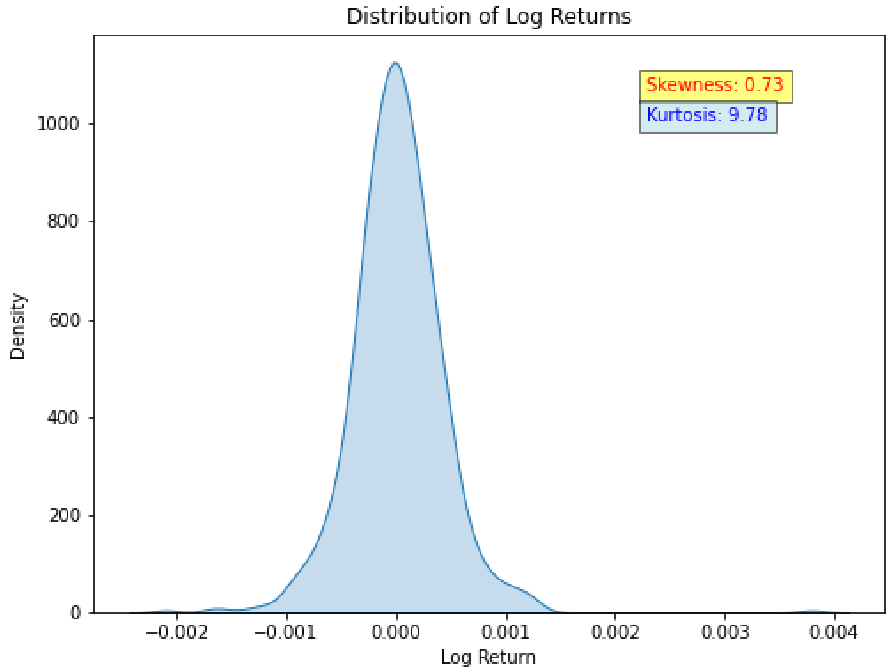

Figure 1. Distribution of log return

3.2. Method

3.2.1. BS Model

According to Black and Scholes [1], the Black-Scholes formula, the value of a European call option on a non-dividend underlying stock is as follow:

$$C(t, S(t)) = S(t)N(d_1) - Ke^{-r(T-t)}N(d_2) \quad (1)$$

Were

$$N(x) := P[Z \leq x] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy$$

$$d_1 = \frac{1}{\sigma\sqrt{T-t}} \left[log(S(t)/K) + (r + \frac{\sigma^2}{2})(T-t) \right]$$

$$d_2 = \frac{1}{\sigma\sqrt{T-t}} \left[log(S(t)/K) + (r - \frac{\sigma^2}{2})(T-t) \right] = d_1 - \sigma\sqrt{T-t}$$

C is the call option price, S0 is the current stock price, K is the option strike price, r is the risk-free interest rate, T is the option expiration time, σ is the volatility of the underlying asset.

Due to the inherent uncertainty and difficulty in forecasting future volatility, the Black-Scholes (BS) model, utilized for option pricing, conventionally assumes the existence of an implied volatility for the underlying stock. This implied volatility is presumed to remain constant over a predetermined period. However, empirical observations reveal that the predictive accuracy of the BS model is often compromised. Deviations have been identified from the assumptions of normal distribution in both price differentials and logarithmic returns. Consequently, it becomes imperative to undertake refinements of the various assumptions underpinning the BS model to achieve a more accurate alignment with market dynamics

3.2.2. GARCH Model

As an enhancement to the ARCH model, the Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model posits that the variance of error terms over a specific period is influenced not only by the variance of the error term with respect to time but also by the past values of the error terms themselves. This model takes into account both lagged values of the disturbance term and lagged values of the conditional variance of the disturbance term. By doing so, the GARCH model addresses the limitation of the ARCH model in capturing volatility persistence, making it a widely employed framework for depicting the volatility of asset returns in financial markets.

If a random volatility follows the process (1) outlined below, it is termed to adhere to a GARCH(p,q) process.

$$a_t = \sigma_t * \epsilon_t \quad (2)$$

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i * a_{t-1}^2 + \sum_{j=1}^{q} \beta_j * \sigma_{t-j}^2 \quad (3)$$

Among them, $\{\epsilon_i\}$is a set of independent and identically distributed random variables with mean 0 and variance 1. At the same time, it is necessary that $\alpha_0 > 0, \alpha_i \geq 0, \beta_j \geq 0,$ and $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1$.

4. Results

4.1. BS Model in Monte Carlo Simulation

In order to facilitate a subsequent comparative analysis of the improvements, the current research initiative initiates by employing the Black-Scholes (BS) model, augmented with implied volatility, to conduct Monte Carlo simulations for the prediction of Carnival Corporation's stock price and the corresponding European call option prices. The simulation commences on July 16, 2023. Spanning a duration of 43 steps, this choice is attributed to the intention of the project to juxtapose the Carnival Corporation's September call option (with a maturity date of September 15) against a backdrop of 43 trading days from the commencement. The risk-free rate utilized is based on the 6-month LIBOR, while the implied volatility is derived from Refinitiv Academic, yielding a value of 0.4988. Presented below are the outcomes of 200 Monte Carlo simulations for the September option with a strike price (figure 2).

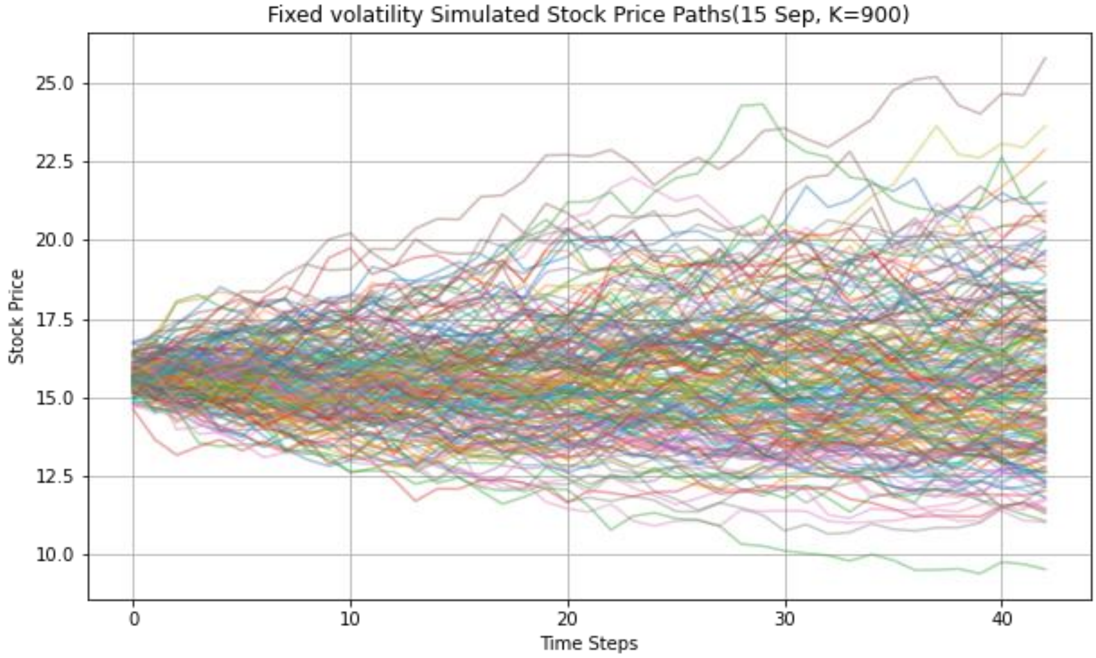

Figure 2. Stock price paths of BS Model

Based on this stock price (table 2), the prices under different strike prices (ranging from 500 to 1600, each option contains 100 stocks) of the call option in September have been obtained.

Table 2. Calculated call option price with multi strike price

Numble	Strike Price(K)	Call Option Price
1	500	1101.29
2	550	1031.93
3	575	1007.54
4	600	977.32
5	625	939.36
6	650	944.61
7	675	913.12
8	700	862.68
9	725	852.13
10	750	850.44
11	775	773.91
12	800	817.43
13	825	779.51
14	850	735.20
15	875	668.09
16	900	673.57
17	950	596.24
18	1000	586.43
19	1200	391.55
20	1600	100.35

4.2. GARCH (1,1) Model in Monte Carlo Simulation

This article uses data from the past three years to fit a GARCH model. Before using the maximum likelihood estimation method to fit the parameters of the GARCH model, the author first checked the auto correlation of the samples. As can be seen from the auto correlation plot below (figure 3), there might be weak but significant autocorrelation when the lag is 5.

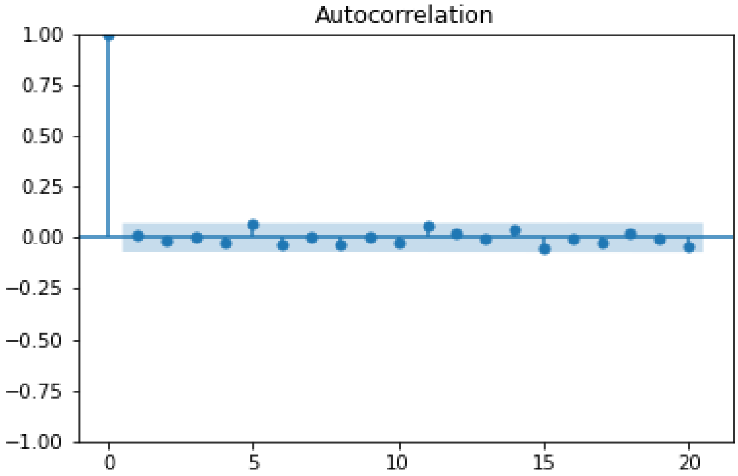

Figure 3. Autocorrelation plot of Carnival historical stock price

Next, according to the results of the GARCH (1,1) model with maxlag 5 fitted by the maximum likelihood estimation method, this project obtained the GARCH equation of the volatility of Carnival's stock price as follows:

$$\sigma_t^2 = 8.22 \times 10^{-5} + 3.03 \times 10^{-2} a_{t-1}^2 + 9.24 \times 10^{-1} \sigma_{t-1}^2$$

$$a_t | r_1, \ldots, r_{t-1} \sim N(0, \sigma_t^2)$$

That is to say, the volatility of each step is related to the volatility of the previous step and a normally distributed random number. In the following Monte Carlo simulation, the paper replaced the implied volatility originally assumed by the BS model with the above-mentioned step-by-step iterative volatility model.

Then, the following stock price distribution and the corresponding expected prices of different strike prices for September call options are obtained as figure 4 and table 3.

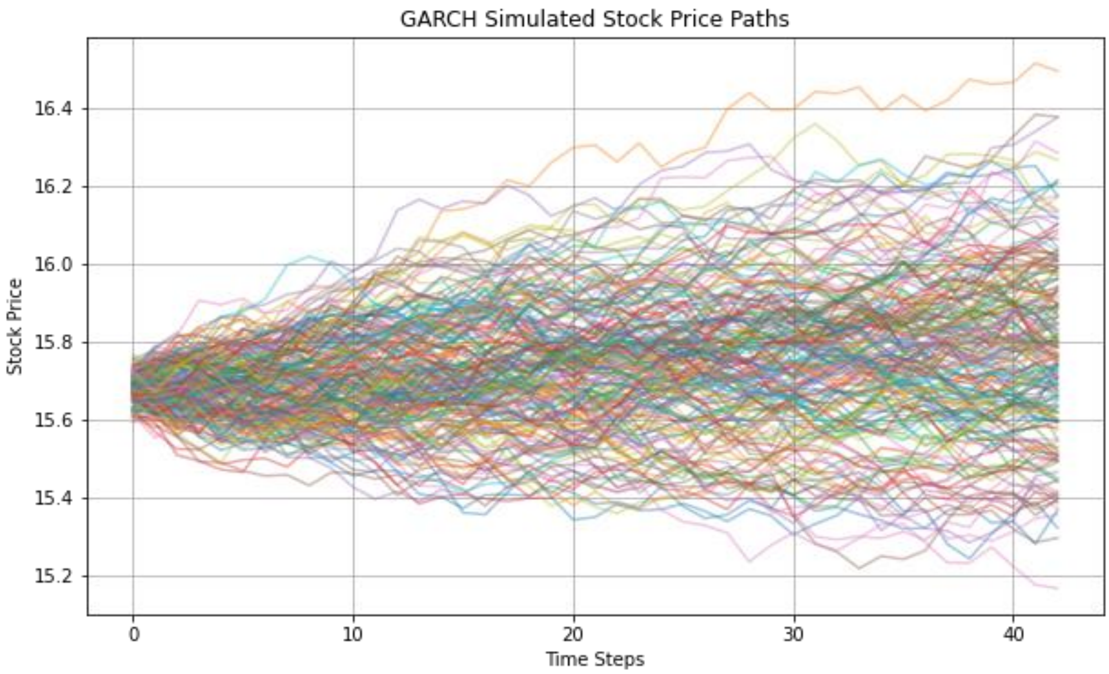

Figure 4. Stock price paths after applying GARCH Model

Table 3. Calculated call option price after applying GARCH Model

Numble	Strike Price(K)	Call Option Price
1	500	1072.25
2	550	1022.63
3	575	995.30
4	600	967.89
5	625	944.69
6	650	922.06
7	675	897.19
8	700	871.24
9	725	848.13
10	750	819.17
11	775	796.29
12	800	774.28
13	825	746.54
14	850	722.49
15	875	695.51
16	900	672.60
17	950	621.90
18	1000	574.68
19	1200	371.06
20	1600	2.12

4.3. Comparison of Empirical Performance of Two Methods

The market price of this series of options can be obtained from Refinitiv. In this project, the RMSE of the above two simulated prices is calculated with respect to the market price, and the results are as table 4.

Table 4. RMSE of the 2 methods comparing with the market performance

	BS with Market	GARCH with Market
RMSE	318.24	308.56

Specifically, the predictive results obtained through the utilization of the GARCH model exhibit a reduction of approximately 3.3% in RMSE when contrasted with the outcomes derived from the Black-Scholes model incorporating implied volatility. Stated differently, the application of the GARCH model enhancements yields option price predictions that are more closely aligned with market prices. This development can be regarded as a demonstrably effective improvement.

5. Discussion

Upon implementing the GARCH model to enhance the underlying assumptions of the Black-Scholes framework, there is a discernible improvement in the predictive accuracy of the corresponding option prices. This enhancement underscores that the volatility captured by the GARCH model, in contrast to a fixed implied volatility, better reflects the true market fluctuations.

On the other hand, the degree of improvement is not particularly pronounced. This can be attributed to the relatively weak autocorrelation present within the sample's volatility. Utilizing the Ljung-Box Q-statistics test, it is evident that the p-values for both maxlag values of 5 and 10 exceed 0.05, indicating the lack of significant autocorrelation in Carnival's stock volatility. Consequently, the autocorrelation effects captured by the GARCH model have not exerted a substantial impact. Another contributing factor is the limitation of this project in not exploring the effects of various GARCH(p,q) models with different p and q. this study relies on the fundamental GARCH(1,1) model.

An additional noteworthy consideration is that the risk-free rate and volatility employed in the Monte Carlo simulations of the BS model were directly sourced from the exchange. In practice, calibration techniques should be employed to derive the implied volatility corresponding to the given risk-free rate. If such an approach were utilized, the simulated option prices generated by the BS model might even outperform the outcomes of the GARCH model enhancements. These are issues that will require further attention and resolution in the future.

6. Conclusion

This study investigates the precision of option pricing by improving the BS model's volatility assumption through the use of the GARCH model. A systematic process of steps that includes Monte Carlo simulations, historical data calibration, and comparative analysis serves as the foundation for the enquiry. The results show that, as compared to the conventional BS model using implied volatility, the GARCH model's incorporation significantly reduces Root Mean Squared Error (RMSE) by about 3.3%. Although noticeable, this improvement is limited by internal constraints. Two important factors can explain this result. First off, the little autocorrelation in the sample data raises concerns about whether applying the GARCH model in this situation is suitable. Furthermore, the implementation of a rudimentary GARCH (1,1) framework fails to fully explore the broader spectrum of GARCH model variations that could potentially yield more substantial enhancements.

The conclusion of this study has significant and wide-ranging ramifications. The results emphasize the significance of adjusting model selection to particular circumstances by highlighting the subtle interplay between model complexity and data characteristics. This effort highlights the necessity for improving assumptions so that the model better captures the market and adds to the larger conversation on improving option pricing accuracy, a fundamental concept in quantitative finance. Additionally, this study serves as a guide for academics and industry professionals attempting to understand the complex world of option pricing and volatility modelling, promoting continued development and breakthroughs in this important area. Exploring a wider variety of GARCH models that are suited to the subtleties of the dataset is suggested as a future direction.

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