Graphical Analysis of the Markowitz Model under Different Constraints

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Abstract. The Markowitz mean-variance model is an important tool in portfolio theory. This study examines the complexity of the model by analyzing its application and performance under different constraints. The study uses Bloomberg data on 10 stocks from 2001 to 2021 to rigorously analyze the impact of limitations on portfolio optimization. The study explores limitations ranging from limiting total allocations and individual weightings to intentional exclusions and long-term strategies to discern the impact on the risk-reward profile. The research uses graphical representations and key metrics such as Sharpe ratios, stock allocation lines, and various portfolio metrics to provide insight into optimal allocation strategies. The results indicate that these constraints have a significant impact on the risk-return profile of a portfolio and that it is essential to take them into account when optimizing a portfolio. The study provides a comprehensive analysis of the Markowitz model and its application under various constraints, which can help investors make informed decisions while optimizing their investment portfolios.

Keywords: Markowitz Model, Portfolio Optimization, Financial Modeling, Constraints, Risk-Return Profile.

1. Introduction

In the field of financial modeling, the Markowitz mean-variance model or Markowitz model is a key construct in the field of portfolio theory. Developed by Nobel laureate Harry Markowitz, this model redefined investment strategy by introducing the key concepts of diversification and the complex relationship between risk and return [1]. The Markowitz model and Modern Portfolio Theory (MPT) laid the foundation for much subsequent research, providing a systematic framework that restructured the way investors construct and optimize their portfolios. In his seminal 1952 paper, Markowitz described a mathematical framework for constructing optimal portfolios taking into account the expected returns, variances, and covariances of different assets [2]. The emergence of this new approach marks a move towards a more systematic and quantitative analysis of investment decisions.

Based on Markowitz's work, many subsequent studies have further expanded the application and influence of Markowitz's model in contemporary financial markets. Based on Markowitz's theory, Sharpe and others independently proposed the concept of capital asset pricing model (CAPM) [3]. This model adds a systematic approach to understanding the relationship between risk and expected return by introducing the concept of beta. Lintner expanded on this work by focusing on the valuation of risky assets and the selection of risky investments in stock portfolios and investment budgeting [4]. Lintner's contribution provides additional insights into the practical application of these models in real-world investment scenarios. Black proposed a new CAPM scheme, also known as zero beta CAPM [5], which provides different calculation methods for assessing balance under risk conditions. These different approaches highlight the evolution and continuous refinement of financial models in response to changing market dynamics.

The Markowitz mean-variance investment model has also been used beyond traditional applications in modern times. Andrew Ang used this model in his book to analyze the stock markets of Japan and the United States. Through the analysis and illustration of the advantages and disadvantages, it further clarified Markowitz's theory of diversified investment [6]. In addition to exploring the further development of Markowitz's theory, Pástor and Stambaugh (1999) also contributed to this discourse by examining the cost of equity and valuation errors in models [7]. His work explores the complexities of social capital and model dynamics, providing insights that will help financial models continue to evolve.

While recognizing the power of the Markowitz model, this study also recognizes that its performance depends on several limitations. These limitations may include, among others, transaction costs, liquidity constraints, and anomalies in asset returns. This article aims to evaluate the performance of the Markowitz model under different constraints, to graphically analyze the causes and effects of these limitations, and to explore potentially stronger investment strategies by analyzing portfolios under different conditions.

The remainder of the paper is organized as follows: Part 2 encompasses introducing data sources, variables' definitions, model specifications, and the research design. Part 3 focuses on data analysis and results using the Markowitz Model with various constraints, featuring graphs and discussions aligned with the paper's main objectives. Finally, Part 4, serves as the conclusion, synthesizing the empirical findings from Part 3 and providing a concise summation of the study's implications within the broader academic context.

2. Methodology

2.1. Data

The data utilized in this study is sourced from Bloomberg, covering the period from 2001 to 2021. The dataset comprises raw daily data for 10 selected stocks: Amazon, Apple, Citrix Systems, JPMorgan Chase & Co., Berkshire Hathaway Inc., The Progressive Corporation, United Parcel Service, FedEx Corporation, J.B. Hunt Transport Services, and Landstar System. The data is processed into 5-day weekly intervals, leading to the computation of prices and returns [8].

Table 1. Summary Statistics

	SPX	AMZN	AAPL	CTXS	JPM	BRK/A	PGR	UPS	FDX	JBHT	LSTR
Annual Average Return	0.08	0.34	0.34	0.16	0.12	0.09	0.15	0.10	0.13	0.23	0.17
Annual StDev	0.15	0.41	0.34	0.42	0.29	0.16	0.21	0.21	0.27	0.31	0.24
beta	1.00	1.35	1.26	1.22	1.36	0.57	0.71	0.83	1.10	1.08	0.80
alpha	0.00	0.24	0.25	0.06	0.02	0.05	0.10	0.04	0.05	0.14	0.11
residual Stdev	0.00	0.36	0.29	0.37	0.21	0.14	0.18	0.18	0.21	0.26	0.21

Table 1 serves as the foundational dataset for constructing graphs and conducting in-depth analyses. It includes key metrics for the 10 stocks, encompassing annual average return, annual standard deviation, beta, alpha, and residual standard deviation. Notably, SPX, representing the Standard & Poor's 500 Index as the benchmark (or the market index) in the table and the model, is expected to have an alpha close to zero, aligning with the assumption of efficient market conditions.

Table 2 provides an overview of the correlation coefficients between the ten stocks and the benchmark index. Each correlation coefficient represents the degree of linear relationship between an individual stock's returns and a benchmark index, providing insights into the degree of closeness or inverse movement of the stock relative to the wider market. This can provide a nuanced understanding of a stock's sensitivity to market movements and the potential diversification benefits in a portfolio.

Table 2. Correlation coefficient matrix

Correlations:	SPX	AMZN	AAPL	CTXS	JPM	BRK/A	PGR	UPS	FDX	JBHT	LSTR
SPX	1.00	0.48	0.54	0.44	0.70	0.52	0.50	0.57	0.61	0.52	0.50
AMZN	0.48	1.00	0.38	0.22	0.25	0.12	0.20	0.30	0.28	0.31	0.26
AAPL	0.54	0.38	1.00	0.33	0.24	0.17	0.24	0.23	0.33	0.27	0.29
CTXS	0.44	0.22	0.33	1.00	0.32	0.18	0.27	0.26	0.33	0.29	0.25
JPM	0.70	0.25	0.24	0.32	1.00	0.45	0.39	0.36	0.44	0.44	0.37
BRK/A	0.52	0.12	0.17	0.18	0.45	1.00	0.26	0.40	0.38	0.24	0.23
PGR	0.50	0.20	0.24	0.27	0.39	0.26	1.00	0.39	0.36	0.28	0.29
UPS	0.57	0.30	0.23	0.26	0.36	0.40	0.39	1.00	0.67	0.46	0.44
FDX	0.61	0.28	0.33	0.33	0.44	0.38	0.36	0.67	1.00	0.54	0.48
JBHT	0.52	0.31	0.27	0.29	0.44	0.24	0.28	0.46	0.54	1.00	0.59
LSTR	0.50	0.26	0.29	0.25	0.37	0.23	0.29	0.44	0.48	0.59	1.00

2.2. Markowitz Model Formulas and Definitions

The Markowitz model, a foundational concept in modern portfolio theory, is employed to optimize the allocation of investments among the selected stocks. The key formulas and definitions include:

-Expected Portfolio Return (EPR):

$$EPR = \sum_{i=1}^{n} w_i \cdot R_i \tag{1}$$

Where w_i is the weight of stock *i* in the portfolio, and R_i is the expected return of stock *i*.

-Portfolio Standard Deviation (PSD):

$$PSD = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i \cdot w_j \cdot \sigma_i \cdot \sigma_j \cdot \rho_{ij}} \tag{2}$$

Where σ_i and σ_j are the standard deviations of stocks *i* and *j* respectively, and ρ_{ij} is the correlation coefficient between stocks *i* and *j*.

-Portfolio Beta:

$$\beta_p = \sum_{i=1}^{n} w_i \cdot \beta_p \tag{3}$$

-Portfolio Alpha:

$$\alpha_p = EPR_p - \left(R_f + \beta_p \cdot \left(EPR_m - R_f\right)\right) \tag{4}$$

where EPR_p is the expected portfolio return, R_f is the risk-free rate, EPR_m is the expected market return.

2.3. Portfolio Metrics and Measures

Several critical portfolio metrics and performance measures are utilized to draw informative graphs in the next part:

-Sharpe Ratio:

$$SR = \frac{EPR_p - R_f}{PSD_p} \tag{5}$$

-Capital Allocation Line (CAL):

$$EPR_p = \frac{EPR_m - R_f}{PSD_m} \cdot PSD_p \quad (6)$$

Minimum Risk Portfolio: this represents the portfolio with the minimum standard deviation, providing the minimum risk achievable in the given set of assets.

-Min Variance Portfolio: this seeks to minimize the overall variance of the portfolio returns.

-Max/Min Return Portfolio: these aim to maximize/minimize the expected portfolio returns.

These metrics, when graphically represented, provide insights into the risk-return profile and optimal allocation strategies for the given set of stocks.

2.4. Constraints

(1) $\sum_{i=1}^{n} |w_i| \leq 2$ the sum of the absolute values of the 10 stocks' weights is constrained to be less than or equal to 2

(2) $|w_i| \leq 1$ the absolute value of each stock's weight is constrained to be less than or equal to 1

(3) Free problem, no constraint here

(4) $w_i \geq 0$ each stock's weight is constrained to be greater than or equal to 0

(5) $w_1 = 0$ the weight for the first stock (SPX) is fixed at zero

3. Results and Discussion

In the pursuit of optimal portfolio allocation among the stocks, the use of constraints is crucial. Levy and Levy ensure pragmatic and realistic portfolio construction using differential constraints based on variance [9]. These constraints not only improve the robustness of the optimization process but also facilitate a nuanced assessment of the merit of including these stocks in a portfolio. This study examines the impact of five different constraints to discern which constraint serves as the superior benchmark for comparing the viability of 10 stocks, thereby providing valuable insights into effective portfolio management strategies.

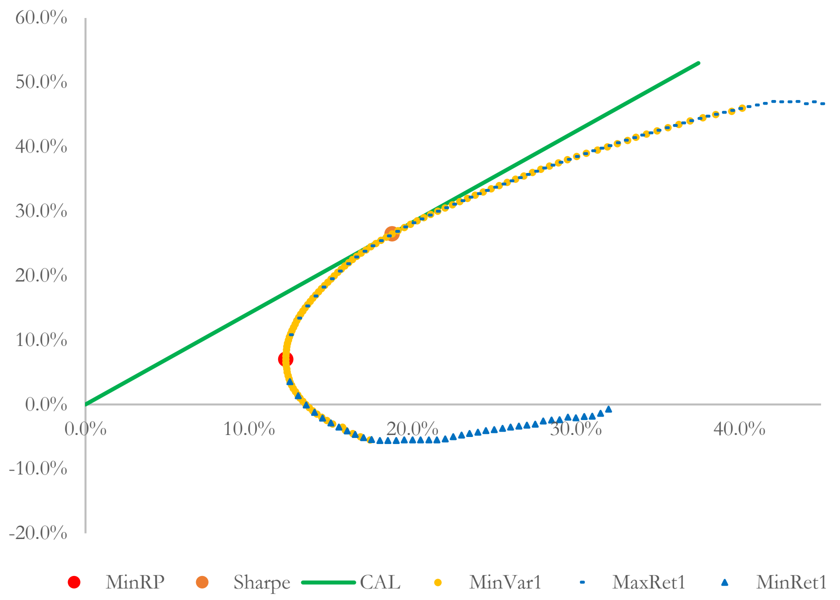

Figure 1. Efficient Frontier with Constraint 1

Photo credit: Original

Figure 1 presents the efficient frontier with the first constraint, which is the sum of the absolute weights, limiting the total allocation of assets in the portfolio, thus introducing a certain degree of control over overall risk. This restriction encourages a diversified portfolio by limiting the sum of absolute weights to a maximum of 2. This limit is considered a risk management tool to ensure that the portfolio is not overly concentrated in one subset of assets. Examination of the efficient frontier concerning this constraint focuses attention on the portfolio for which the sum of the absolute weights is maximized within specific limits. These portfolios represent the optimal compromise between risk and return while maintaining diversification.

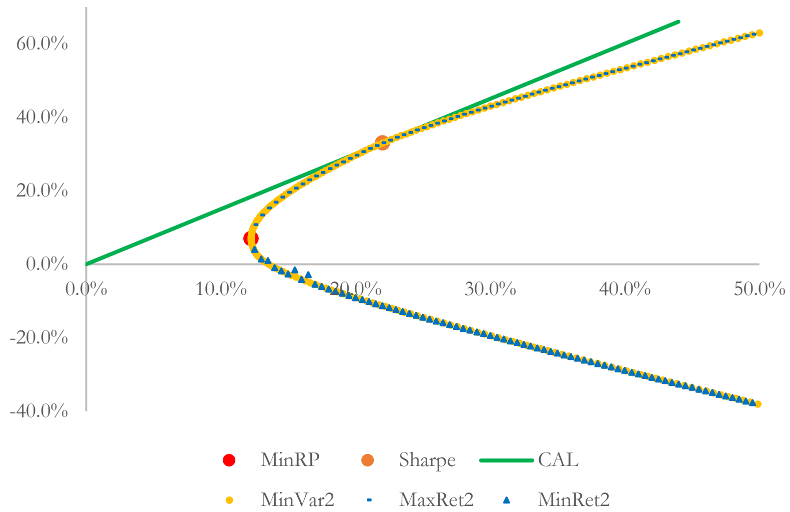

Figure 2. Efficient Frontier with Constraint 2

Photo credit: Original

Figure 2 presents the efficient frontier with the second constraint, individual absolute weights, which prevents extreme concentration on a single asset in the portfolio. By establishing that a single weighting cannot exceed 1, this restriction promotes a balanced and diversified allocation, protecting against the risk of over-reliance on specific securities and encouraging a more resilient portfolio. Compliance with this restriction is essential for investors who wish to monitor the impact of changes in individual assets on the entire portfolio. An efficient frontier analysis involves identifying the portfolio in which the weight of each asset reaches a maximum value of 1 without violating the constraints. These portfolios demonstrate a diversified portfolio that spreads risk across a variety of assets while avoiding overemphasis on a single security.

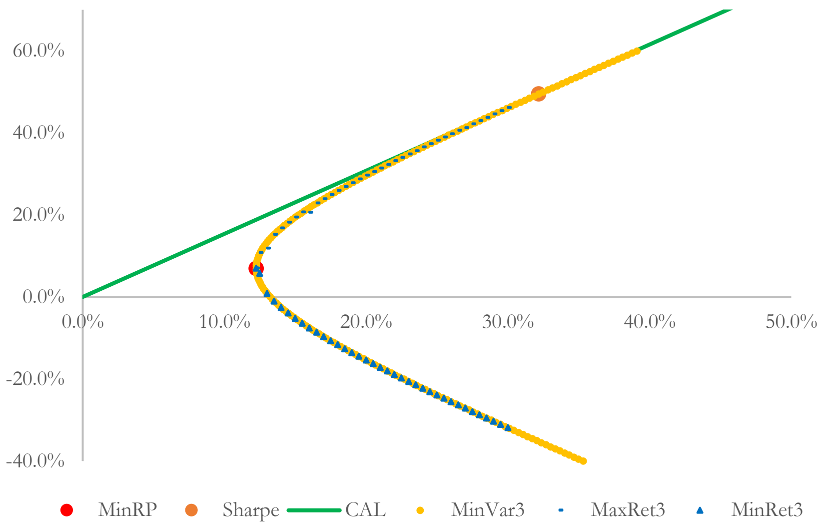

Figure 3. Efficient Frontier with no Constraint

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Figure 3 shows the free problem, no specific constraints are imposed beyond the standard ones of the Markowitz model, and this article highlights that investors benefit from some flexibility in constructing their portfolios. Since there are no further restrictions, risk-return scenarios can be explored extensively. However, investors must interpret this freedom as an opportunity and a challenge, carefully considering real-world constraints, transaction costs, and liquidity. Efficient frontier portfolios, without specific constraints, allow us to understand the maximum risk-return combination achievable in certain market conditions. However, investors should evaluate whether these portfolios are consistent with real-world investment considerations, taking into account transaction costs, market liquidity, and other real-world limitations. Looking at the efficiency frontier helps investors find a compromise between theoretical optimization and practical feasibility.

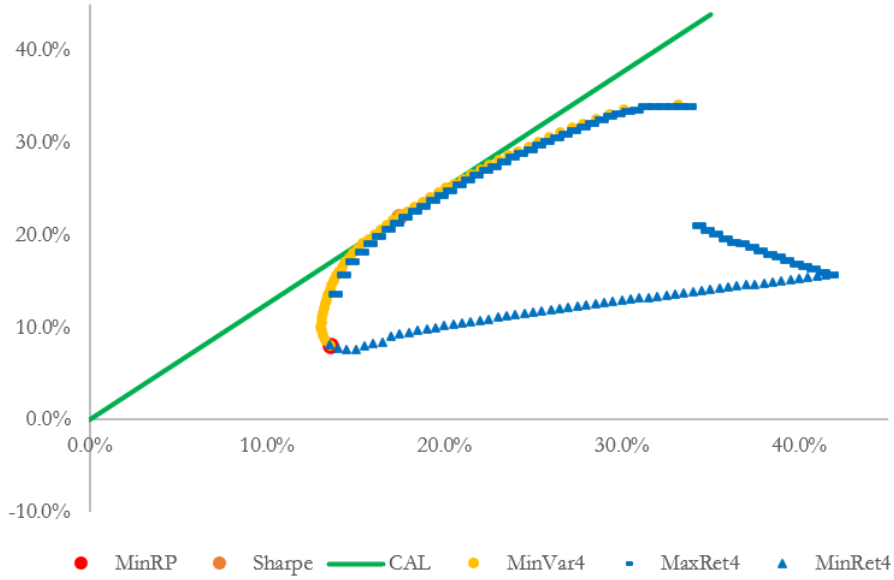

Figure 4. Efficient Frontier with Constraint 4

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As shown in Figure 4, the non-negative weight constraint ensures that all asset weights are greater than or equal to 0, reflecting a fully long portfolio strategy. This restriction is consistent with traditional approaches that do not allow short selling and emphasize active exposure to selected assets. This limit is interpreted as a commitment to build a portfolio entirely composed of long positions, thus simplifying the execution of investment decisions. In most portfolio optimization models, the non-negative weight constraint is a fundamental condition observed at all points of the efficient frontier. This finding reinforces the commitment to long-term strategies, and investors can rely on the efficient frontier to guide them in selecting portfolios that meet this key criterion.

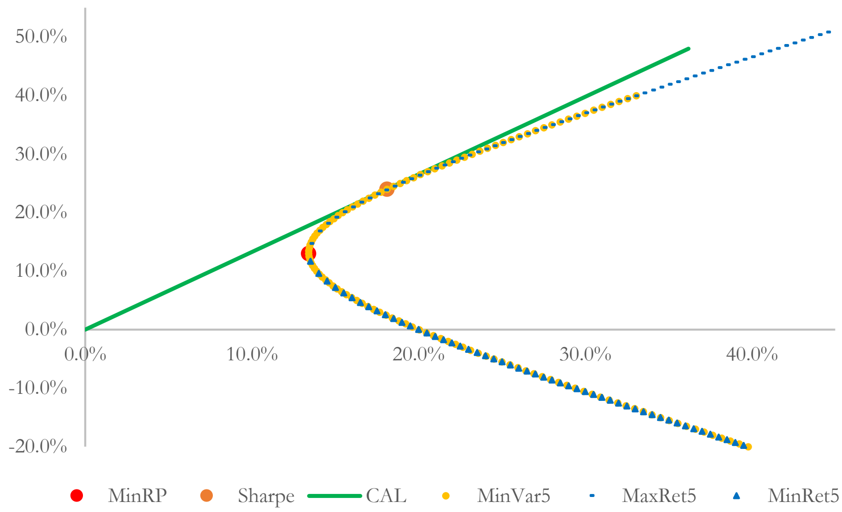

Figure 5. Efficient Frontier with Constraint 5

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Referring to Figure 5, Setting the weight of the first portfolio value (SPX) to zero implies a deliberate choice to exclude investing in the stock market index. This restriction is seen as a decision to avoid exposure to the broader market volatility represented by the SPX and may indicate a preference for a more concentrated or specialized portfolio strategy. In the context of this restriction, the efficient frontier presents a portfolio in which the SPX weight is always zero. This intentional exclusion may result in a portfolio that is less correlated to broader market movements, which may offer a unique risk-return profile. Investors who wish to construct portfolios that are less sensitive to market fluctuations can identify and price them at the efficient frontier [10].

4. Conclusion

In summary, this article provides a comprehensive analysis of the Markowitz mean-variance model in portfolio optimization under various constraints. Constraints play a key role in formulating optimal investment portfolios and influencing risk management and diversification strategies. The results suggest that imposing limits on aggregate allocations, individual weightings, or exclusions from market indices requires investors to carefully consider real-world constraints such as transaction costs and liquidity. Effective frontier analysis demonstrates the tradeoffs between risk and return, guiding investors toward selecting portfolios that match their risk appetite and investment objectives.

The comparison of constraints highlights the importance of tailored approaches in portfolio construction, highlighting the need for a balance between theoretical optimization and practical feasibility. This illustrates the critical importance of limits that encourage diversification and risk control, revealing their impact on portfolio dynamics. This study not only contributes to advancing the current debate on portfolio theory but also provides practitioners with practical insights into effective portfolio management strategies.

Furthermore, this article demonstrates the adaptability of the Markowitz model in a dynamic financial environment where market conditions are constantly changing. This article demonstrates the resilience and relevance of the Markowitz model in portfolio optimization, making it a guiding framework for investors and financial analysts. The results demonstrate that Markowitz's model and theory can help investors and ensure an intelligent balance between risk and return in investment decisions. It is necessary to understand its limitations and adopt a tailored approach to balance theoretical optimization and practical feasibility for effective portfolio management.

In conclusion, this article provides a nuanced understanding of the portfolio optimization performance of the Markowitz mean-variance model under different constraints. The results show that constraints influence the formulation of optimal portfolios and influence risk management and diversification strategies. The comparison of constraints highlights the importance of tailored approaches in portfolio construction, highlighting the need for a balance between theoretical optimization and practical feasibility. Finally, we conclude that effective portfolio management requires an understanding of limitations and a tailored approach that balances theoretical optimization and practical feasibility.

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