A Comparative Study on Jorion's Estimator in Contemporary Market

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Abstract. In knowledge of the prominence of Markowitz model in modern portfolio theory, and the following controversy on the appropriate choices of input parameters, this study examines the practical performance of Jorion's estimator as one of the estimation strategies to reduce the estimation error inherent in traditional input parameter. A comprehensive dataset of real stock market returns, encompassing 7 assets and 2 industries, is employed. Jorion's estimator entails the adjustment of the mean of sample return towards the mean of minimum variance portfolio. The results are then integrated into the optimization phase of Markowitz model. To evaluate the effectiveness of Jorion's estimator, realized returns and return volatility of Markowitz model adapted by Jorion's estimator are compared against the traditional model. The findings provide insights into the practicability of Jorion's estimator for portfolio management in contemporary capital market. After comparisons, all 3 measures of realized returns from Jorion's estimator outperforms the counterpart from traditional parameter. This study affirms that the classic sample mean is inadequate as an estimator for future performance, and sophisticated techniques, such as Jorion's estimator should be employed in real-world investment strategies.

Keywords: Markowitz Mean-Variance Model, Optimal Portfolio Weights, Shrinkage Estimator.

1. Introduction

The process of portfolio design often entails dividing the entire portfolio into risk-free and risky assets, as well as determining the composition of the risky portion [1]. One of the fundamental assertions in modern financial literature posits that investors exhibit risk aversion and are inclined to assume more significant risks only when a commensurate return increase is expected [2]. The assumption as mentioned earlier has underscored the importance of optimizing portfolios, as it is required to increase returns and decrease risks simultaneously. Following the second assumption, which posits that investment decisions are influenced solely by returns and risks, investment returns should be evaluated through the percentage change in portfolio value over a given holding period (Holding-Period Returns), and that risk measured by the standard deviation (σ) of the return rate. Therefore, the remaining problem for an optimised portfolio is to ascertain the composition of the portfolio that yields the Highest Holding-Period Returns and minimized standard deviation (σ).

The mean-variance model, initially introduced by Harry Markowitz in 1952, was contrived to solve this problem. The calculation of Markowitz's formula necessitates knowledge of estimated returns (μ_i), standard deviation (σ_i) for each asset i, and the correlation coefficient (ρ_{ij}) between assets (i, j) [3]. Consequently, the optimised portfolio weight of each asset is determined when the expected utility E(U), whose function should be hypothesized in the first place, is maximised [4].

Despite the pivotal role of the Markowitz model in the domain of asset management, its practical efficacy has long been controversial. The prevalent approach in Markowitz model practice is to substitute the estimated returns (μ_i) of the asset (i) with the mean value of return rates ($\hat{\mu}_i$) during the sample period without considering its inherent uncertainty. According to [5] and [6], the estimation error in expected returns is the most influential in rendering extreme portfolio weights. Due to the accumulated estimation error, the solution of portfolio weights often gives "an unbalanced asset allocation and a lack of diversification" [7]. Therefore, many scholars have advocated for using more suitable estimators that are less subject to estimation error [8].

As “one of the most prominent extensions of MV (mean-variance model)” [3] Jorion (1985) derived the shrinkage estimator in an attempt to reduce estimation error in historical means [9]. It is argued that shrinking the historical mean of individual asset return towards a standard value can mitigate the impact of estimation error and leading to better performance in ex-post periods. The efficacy of such an estimator in determining a consistent and improved portfolio was evident in Jorion’s simulation study and the following capital market study.

This article draws on the conclusion of Jorion and attempts to investigate the efficacy of Jorion’s estimator in the capital market during the 2010s. The rest of the paper is organised as follows. In Section 2, the mathematical framework of Markowitz Mean-Variance Model and Jorion’s estimator will be revisited and explained in depth. In Section 3, the selected data will be described, and the process of model construction will be explained. In Section 4, the result of the capital market study will be presented. Finally, this article will be concluded in Section 5.

2. MV model and jorion’s estimator

The original Markowitz model considers the market with N risky assets, and the optimisation problem concerns the capital allocation among risky assets. Based on the assumption that investors have a quadratic utility function and that returns are normally distributed, the optimisation problem is simplified to the maximisation of certainty equivalent ϕ, where:

$$\phi = \boldsymbol{\omega}'\boldsymbol{\mu} - \left(\frac{\lambda}{2}\right)\cdot\boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega}, \text{ s.t } \sum_{i=1}^{N} w_i - 1 = 0 \tag{1}$$

In this equation, $\boldsymbol{\omega}=(w_1, w_2, \ldots w_N)'$ is an N×1 vector representing weights of each risky asset in the portfolio, $\boldsymbol{\mu}$ is an N×1 vector representing the expected return of each asset, $\boldsymbol{\Sigma}$ is an N×N variance-covariance matrix, $\mathbf{i}$ is the N×1 vector of 1 and λ is the risk aversion coefficient. Using the Lagrangian function, the maximisation of ϕ is transformed to:

$$L = \boldsymbol{\omega}'\boldsymbol{\mu} - \left(\frac{\lambda}{2}\right)\cdot\boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega} + \theta(\boldsymbol{\omega}'\mathbf{i} - 1) \tag{2}$$

Where θ denotes the Lagrange multipliers. The first-order condition of L is:

$$\boldsymbol{\mu} - \lambda\boldsymbol{\Sigma\omega} + \theta\mathbf{i} = 0 \tag{3}$$

After appropriate substitution,

$$\boldsymbol{\omega} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{i}}{\mathbf{i}'\boldsymbol{\Sigma}^{-1}\mathbf{i}} + \left(\frac{1}{\lambda}\right)\cdot\left(\boldsymbol{\Sigma}^{-1} - \frac{\boldsymbol{\Sigma}^{-1}\mathbf{i}\mathbf{i}'\boldsymbol{\Sigma}^{-1}}{\mathbf{i}'\boldsymbol{\Sigma}^{-1}\mathbf{i}}\right)\boldsymbol{\mu} \tag{4}$$

If similar steps are conducted, except for changing the maximisation problem to the minimisation of portfolio variance, the weights for global minimum variance portfolio (GMV) can be derived, where

$$\boldsymbol{\omega}_{\mathbf{GMV}} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{i}}{\mathbf{i}'\boldsymbol{\Sigma}^{-1}\mathbf{i}}. \tag{5}$$

It immediately follows that the optimal portfolio weights in terms of expected utility are a combination of the GMV weights and another factor depending on the covariance and return estimators. In the classic MV model, the sample mean $\hat{\boldsymbol{\mu}}$ and the variance-covariance matrix using sample mean as calculation input are used to replace and $\boldsymbol{\Sigma}$ in the formulas mentioned earlier, respectively. However, [10] proved that $\hat{\boldsymbol{\omega}}_{\mathbf{GMV}}$ is an unbiased estimator of $\boldsymbol{\omega}_{\mathbf{GMV}}$ and the difference between the estimated weight of each asset and actual weight follows a scaled t-distribution. On the contrary, $\hat{\boldsymbol{\omega}}$ in the utility maximisation equation is a biased estimator. Therefore, in order to derive portfolio weights for utility maximisation with more reliability, it is sensible that more importance should be placed on the GMV portfolio.

In addition, the simple practice of replacing $\boldsymbol{\mu}$ with sample mean has long been subject to criticism. According to [6], even with a strong constraint of all weights being positive, minor variations in return estimators can result in a shake-up of portfolio weights, leading to poor reliability of results. Specifically, the estimation error of means is 21 times as damaging as covariance and 11 times as destructive as variance. These findings empirically proved the conclusion of [11] that the historical mean is an inadmissible estimator for a portfolio with the number of assets greater than 2.

Jorion's shrinkage estimator was conceived to mitigate the uncertainty in return rates and the vulnerability of portfolio weights. Drawing on James and Stein's suggestion that the appropriate return estimator should be the weighted average of the sample mean and a standard value, Jorion proposed that the standard value should be the return of the GMV portfolio. The weight for computation should be an equation decided by the difference between sample means of assets and return of GMV portfolio, number of assets, and observations in the sample periods (denoted by T). Specifically:

$$\boldsymbol{\mu}_{\mathbf{J}} = \alpha \cdot \boldsymbol{\mu}_{\mathbf{GMV}} + (1-\alpha)\hat{\boldsymbol{\mu}} \tag{6}$$

Where $\alpha = \frac{N-2}{T-N+2} \cdot \frac{1}{(\hat{\boldsymbol{\mu}}-\boldsymbol{\mu}_{\mathbf{GMV}})'\hat{\boldsymbol{\Sigma}}^{-1}(\hat{\boldsymbol{\mu}}-\boldsymbol{\mu}_{\mathbf{GMV}})}$ [9]

For the subsequent computation of the utility maximization portfolio, $\boldsymbol{\mu}_{\mathbf{J}}$ and $\hat{\boldsymbol{\Sigma}}$ are used as input parameters.

3. Data description

The primary objective of this study is to evaluate the efficacy of Jorion's estimator. Consequently, it is imperative to compare the outcomes obtained from both the conventional MV model and the MV model that has been modified using Jorion's methodology. An ex-post study is undertaken to evaluate the usefulness of Jorion's estimator, which is known for its ability to improve the consistency of portfolio weights and enhance performance in future periods. Initially, the optimized portfolio weights for both approaches are computed throughout the specified sample periods. Furthermore, in practical investing scenarios, investors rely on past data to inform their decision-making processes, despite the fact that portfolio gains are only realized in subsequent periods. To replicate this particular situation, the out-of-sample returns are computed by employing the optimal portfolio weights obtained in the initial phase, and the returns observed one period subsequent to the sample period. To provide precise details, the initial sample period T is defined as the first month through the 36 month of the entire time under consideration. Hence, the ex-post returns computation employs the returns seen in the 37th period, along with the portfolio weights calculated from the sample period T. For the purpose of this study, a dataset consisting of price data from seven stocks has been chosen. These stocks are Adobe Inc. (coded as ADBE), International Business Machines Corporation (IBM), SAP SE (SAP), Bank of America Corporation (BAC), Citigroup Inc. (C), Wells Fargo & Company (WFC), and The Travelers Company Inc. (TRV). These companies are engaged in operations within the technology and financial service industries. The total returns data, which has been adjusted for dividend payouts and stock splits, for the seven chosen assets from December 31, 2004, to December 31, 2014, has been collected from Yahoo! Finance. To mitigate the non-Gaussian distortions inherent in daily data, return rates are computed on a monthly basis, a practice commonly accepted as conforming to a normal distribution [10]. As a result, there exist a total of 120 monthly return rates throughout a span of 10 years.

The maximizing problem and the construction of the GMV portfolio are performed utilizing Excel, specifically employing the Visual Basics Editor for calculating the shrinkage weight and the Solver Add-ins for determining the ideal portfolio weights. No additional constraints were imposed, other than $\sum_{i=1}^{N} w_i - 1 = 0$, in the configuration of Solver. The risk aversion coefficient is assigned a value of 2, whose actual value is estimated to be slightly larger than 1 [12]. Hence, the task of maximizing the certainty equivalent is subsequently transformed into maximizing the difference between expected return and expected portfolio variance.

4. Results

The ex-post analysis was performed on the aforementioned dataset using both the traditional MV model and the MV model adapted by Jorion's estimator. The findings of this analysis are provided in Table 1.

Table 1. Realised return based on a 36-month sample

	Realized returns based on 36-month samples		
	mean	Standard deviation	Sharpe Ratio
Traditional MV	4.201%	44.161%	9.513%
Jorion’s Estimator	5.257%	18.355%	28.639

Based on a dataset of 120 observations for each asset spanning from January 31, 2005, to December 31, 2014, a total of 84 pairs of outcomes have been computed. The initial 36-month period serves as the primary sample period for determining the first set of optimal weights of the portfolio. 84 pairs of realized returns are computed based on the return rates of seven assets, spanning from January 31, 2008, to December 31, 2014. Table 1 presents statistical measures pertaining to the values of realized returns. Specifically, it includes the mean, standard deviation, and the Sharpe ratio, which is determined as the ratio of the mean to the standard deviation. The inclusion of the Sharpe ratio in the analysis was motivated by its widely recognized status as a suitable metric for evaluating portfolio performance. The data presented in the table demonstrates that Jorion's Estimator has superior performance compared to the standard MV methodology across all dimensions. Despite the mean return rates exhibit a marginal distinction of around 1%, the cumulative effect over the whole seven-year duration results in an additional return of 89%. The observed disparity in standard deviation, a commonly used metric for assessing risk, was notable. Specifically, the traditional mean-variance (MV) model yielded a standard deviation that was 2.4 times greater than that of the portfolios adapted using shrinkage techniques. The substantial disparity in standard deviations results in a disadvantageous evaluation of the Sharpe ratio in the traditional mean-variance model. The Sharpe ratio of the classic mean-variance (MV) model is just one-third of the Sharpe ratio obtained by Jorion's estimator. In a straightforward interpretation, it can be observed that the optimal portfolio weights derived from sample means failed to effectively account for the trade-off between increased risks and better returns. Conversely, portfolios that were modified towards the Global Minimum Variance (GMV) portfolios demonstrated the ability to maintain risk within a tolerable range.

The efficacy of Jorion's estimator can be elucidated by the scale of the shrinkage weights. The mean of the 84 estimated shrinkage weights is 0.74, with a maximum value of 0.94, a minimum value of 0.44, and the majority of weights (70.23%) exceeding 0.7. According to Jorion's findings [9], when the magnitude of the shrinkage weights grows, the efficient frontier will become progressively more horizontal, resulting in a reduction of the gap between the efficient frontier and the horizontal line indicating $\boldsymbol{\mu}_{\mathbf{GMV}}$. Consequently, the portfolio weights that yield the highest level of efficiency will gradually converge towards $\boldsymbol{\omega}_{\mathbf{GMV}}$. Furthermore, as demonstrated in Section 2, it has been established that $\boldsymbol{\omega}_{\mathbf{GMV}}$ is solely governed by the sample variance-covariance matrix. Hence, it can be inferred that the imposition of constraints by the GMV portfolio aims to mitigate the occurrence of extreme values or significant fluctuations in the optimal portfolio weights across different time periods.

This proposition is supported by the data presented in Figure 1 and Figure 2. These figures depict the weights of two prominent assets, IBM and Citigroup, throughout 84 time points in portfolios. It is evident that the weights assigned to both assets in Jorion's technique exhibit a gradual smoothing pattern over time, with occasional oscillations observed during the initial and final periods. On the contrary, the weights assigned to both assets as calculated by traditional mean-variance (MV) models displayed unpredictable fluctuations. Using IBM as an example, the initial weights of the company can be observed to commence at approximately 70%. Over a period of six months, these weights might increase significantly, reaching a peak of 340%. However, it is noteworthy that these weights can subsequently decline to a minimum value of -660%. The considerable fluctuations in the composition of the portfolio may perhaps account for the limited predictive accuracy observed in the ex-post analysis.

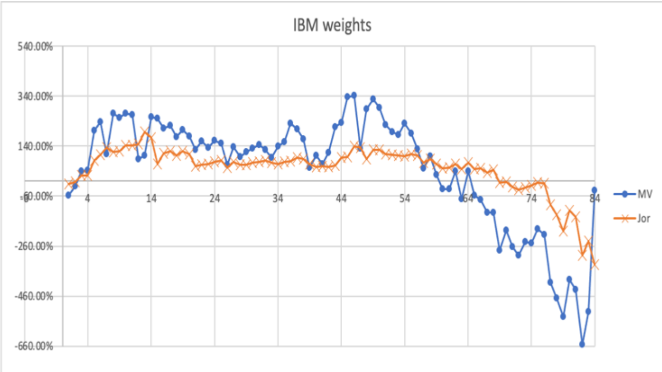

Figure 1. IBM weights in optimal portfolio

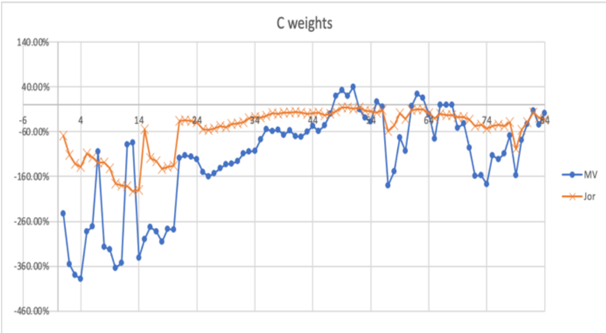

Figure 2. Citigroup weights in optimal portfolio

5. Conclusion

The present study examined the out-of-sample comparative performance of the traditional mean-variance (MV) model, which employs the sample mean as an estimate, and the modified model that incorporates Jorion's estimator. This study utilizes real stock data from seven firms, spanning the time period from January 31, 2005 to December 31, 2014, for computational analysis. The optimization issue is formulated as the maximization of certainty equivalents for a quadratic utility function, where the risk aversion coefficient is set to 2. The findings from the ex-post analysis demonstrate that the portfolios identified using Jorion's estimator are characterized by lower levels of risk, as evidenced by the reduced volatility observed in the subsequent returns, in compared to the portfolios generated by the traditional mean-variance (MV) model. The estimation of GMV portfolio weights holds significant importance in Jorion's estimator, as it contributes to enhanced consistency in portfolio weights across time. Hence, it can be inferred that Jorion's estimator demonstrates effectiveness in the 2010s in terms of minimizing estimation error in utility maximization portfolios.

Nevertheless, it is important to acknowledge that this study is not without its limits. One primary limitation of this study is the subjective selection of the 7 assets that comprise the complete dataset. The selecting method does not employ any random sampling techniques. Hence, the capacity to reproduce the findings of this study is hindered. Additional research that incorporates a broader range of resources could be conducted in order to improve the overall applicability of the findings. Furthermore, it should be noted that the scope of this study is predominantly constrained. According to Bessler et al., there exist many approaches to determine the optimal portfolio, apart from the expected utility maximization. These alternative methods encompass the maximum Sharpe ratio portfolio and the tangency portfolio. Additional research should be undertaken to analyze the relative performance of Jorion's estimator in portfolios that incorporate risk-free assets, specifically in the context of tangency portfolios.

In general, this study confirms that the utilization of the sample mean as an estimator for future returns is inadequate. As a result, it is recommended that professionals in the industry employ more advanced techniques, such as Jorion's estimator, while applying the Markowitz MV model.

References

[1] Z. Bodie, A. Kane, and A. J. Marcus, "Capital Allocation to Risky Assets," Investments, 12th, New York, US: McGraw-Hill Education, 167-200 (2021).

[2] E. Vercher, J. D. Bermúdez, and J. V. Segura, "Fuzzy portfolio optimization under Downside Risk Measures," Fuzzy Sets and Systems, 158, 769–782 (2006).

[3] W. Bessler, H. Opfer, and D. Wolff, "Multi-asset portfolio optimization and out-of-sample performance: An evaluation of black–litterman, mean-variance, and naïve diversification approaches," The European Journal of Finance, 23, 1–30 (2014).

[4] J. Knight and S. E. Satchell, "Exact properties of measures of optimal investment for benchmarked portfolios," Quantitative Finance, 10, 495–502 (2009).

[5] M. J. Best and R. R. Grauer, "Sensitivity analysis for mean-variance portfolio problems," Management Science, 37, 980–989 (1991).

[6] V. Kumar. Chopra and W. T. Ziemba, "The effect of errors in means, variances, and covariances on Optimal Portfolio Choice," The Journal of Portfolio Management, 19, 6–11 (1993).

[7] F. Becker, M. Gürtler, and M. Hibbeln, "Markowitz versus Michaud: Portfolio optimization strategies reconsidered," The European Journal of Finance, 21, 269–291 (2013).

[8] J.-R. Yu, W.-J. Paul Chiou, W.-Y. Lee, and S.-J. Lin, "Portfolio models with return forecasting and transaction costs," International Review of Economics & Finance, 66, 118–130 (2020).

[9] P. Jorion, "International portfolio diversification with estimation risk," The Journal of Business, 58, 259-278 (1985). doi:10.1086/296296.

[10] Y. Okhrin and W. Schmid, "Distributional properties of portfolio weights," Journal of Econometrics, 134, 235–256 (2006).

[11] W. James and C. Stein, "Estimation with quadratic loss," Springer Series in Statistics, 443–460 (1992).

[12] R. Layard, G. Mayraz, and S. Nickell, "The marginal utility of income," Journal of Public Economics, 92, 1846–1857 (2008).