A New CKLS Model with GARCH-type Volatility and SSAEPD Error Terms

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Abstract. This paper introduces a novel extension to the Chan, Karolyi, Longstaff, and Sanders (CKLS) model by incorporating GARCH-type volatility and replacing traditional error terms with Standardized Standard Asymmetric Exponential Power Distribution (SSAEPD) error terms. The classic CKLS model, while robust, fails to account for certain empirical irregularities in financial data such as skewness and fat tails. Our proposed CKLS-GARCH_SSAEPD model aims to address these limitations by integrating the GARCH mechanism for modeling time-varying volatility, as pioneered by Bollerslev (1986), and adopting SSAEPD for error distribution to capture the asymmetry and leptokurtosis in financial datasets. We reevaluate the robustness of the CKLS model using two distinct samples of one-month US Treasury bill yields spanning from 1964 to 2013. The empirical analysis is conducted using Maximum Likelihood Estimation (MLE), with model comparisons based on the Akaike Information Criterion (AIC). Supplementary statistical tests, including the Ljung-Box, Kolmogorov-Smirnov, and Likelihood Ratio tests, are applied to assess the model's fit and robustness. Our findings suggest that the CKLS-GARCH_SSAEPD model provides a better fit to the data compared to the traditional CKLS model, reflecting enhanced capability in capturing the dynamics of interest rate movements and volatility. The modifications introduced in this model significantly improve the understanding and forecasting of interest rates, thereby offering valuable insights for risk management and financial econometrics.

Keywords: CKLS Model; GARCH Volatility; SSAEPD Error Terms; Interest Rate Modeling; Financial Econometrics.

1. Introduction

Interest rate is a crucial economic variable for risk management, asset pricing, and return forecasting. Numerous models have been proposed to elucidate the behavior of interest rates. Among these, one-factor interest rate models are a prominent category used in the pricing of interest rate derivatives. Various types of one-factor interest rate models exist, as detailed in Table 1. The Ornstein-Uhlenbeck process has been employed to derive the equilibrium model of bond prices [1]. The square root process was used to suggest a single-factor structural model [2]. Additionally, a comprehensive comparison and generalization of all one-factor models into the CKLS model was conducted, representing a broad class of single-factor models with unrestricted power [3].

Table 1. Interest Rate Models.

Authors	Models
Merton (1973)	GBM
Cox and Ross (1976)	CEV
Vasicek (1977)	Vasicek
Dothan (1978)	Dothan
Cox et.al. (1985)	CIR, SR
Chan et.al. (1992)	CKLS
Baz and Das (1996)	Jump-Diffusion
Singleton (2002)	Affine
Bernadell et.al. (2005)	BCN

A substantial body of empirical research focuses on CKLS models. The flexibility of the CKLS model allows for a better explanation of interest rate characteristics. Nonetheless, the CKLS model has its limitations. For instance, it has been criticized for overlooking the possibility of non-linear drift [4], and for lacking a second stochastic factor [5]. Furthermore, empirical tests on various three-factor models have shown that none are capable of capturing the non-Gaussian features of observed data [6]. In response, several researchers have extended the CKLS model in different ways, as shown in Table 2. Notably, stochastic volatility was introduced as an additional factor, proposing a two-factor CKLS model [5], and the CKLS_AEPD model was suggested to accommodate the non-normal characteristics of financial data [7].

Table 2. Extensions for CKLS model

Authors	Models
Baz and Das (1996)	CKLS-Jump
Jegadeesh and Pennacchi (1996)	Two-factor CKLS (with mean reversion)
Andersen and Lund (1997)	Two-factor CKLS (with volatility)
Ball and Torus (1999)	Two-factor CKLS (with volatility)
Liu and Zheng (2006)	Markov Regime Switching CKLS
Li et.al. (2011)	CKLS_AEPD
Sorwar (2011)	CKLS with non-linear drift

Building on prior research, this paper proposes a new CKLS model incorporating GARCH-type volatility as described by Bollerslev [8] and modifying error terms to Standardised Standard Asymmetric Exponential Power Distribution (SSAEPD). The concept of AEPD, first introduced by

Zhu and Zinde-Walsh [9], extends the Normal distribution to include both skewness and tail parameters, capturing the fat tails and asymmetric effects prevalent in financial data (see Table 3).

Table 3. Extensions and Applications of the Normal Distribution.

Authors	**Distributions and their Applications**
De Moivre (1738)	Normal distribution
Gauss (1809)	Normal applied in astronomy
Subbotin (1923)	EPD
Aitchison J. and Brown J.A.C. (1957)	Lognormal distribution
Azzalini (1986)	SEPD
Zolotarev V.M. (1986)	Stable distribution
Bollerslev (1987)	Student-t distribution
Fernandez et al. (1995)	Modified SEPD
Swamee P.K. (2002)	Near Lognormal distribution
Ayebo and Kozubowski (2004)	SEPD in finance
DiCiccio and Monti (2004)	Properties of MLE of the SEPD
Zhu and Zinde-Walsh (2009)	AEPD

Notes: EPD=Exponential Power Distribution; SEPD=Skewed Exponential Power Distribution; AEPD=Asymmetric Exponential Power Distribution.

This paper introduces the CKLS-GARCH_SSAEPD model. We reassess the robustness of the results from Chan et al. [3] by testing several hypotheses: 1) the sensitivity of the conditional volatility of the process to interest rates, 2) the existence of a structural shift in the CKLS model, and 3) the performance of our new model with non-normal error terms and GARCH-type volatility compared to the classic CKLS model.

To address these questions, we analyze the one-month US Treasury bill yield across two samples. Sample A spans from June 1964 to December 1989, consistent with the data used in Chan et al. [3], while Sample B extends from January 1990 to December 2013. All data were sourced from the Center for Research in Security Prices (CRSP). The Maximum Likelihood Estimation (MLE) method is utilized for parameter estimation, and models are compared using the Akaike Information Criterion (AIC). The Ljung-Box test, Kolmogorov-Smirnov (KS) test, and Likelihood Ratio (LR) test are applied in the empirical analysis.

Empirical findings indicate that in our new model, interest rate volatility is not sensitive to the level of short-term yield during both periods. However, our model does display mean reversion, a characteristic of the one-factor CKLS model. The estimates of the SSAEPD parameters effectively capture skewness, asymmetry, and fat-tail characteristics in our sample. According to the AIC, our new model with SSAEPD error terms and GARCH-type volatility demonstrates a superior in-sample fit compared to the classic CKLS model.

The structure of this paper is outlined as follows: Section 2 discusses the model and methodology, Section 3 is dedicated to simulation analysis, Section 4 presents the data and empirical results, and Section 5 concludes with discussions on future extensions.

2. Model and Methodology

2.1. The CKLS-GARCH SSAEPD Model

The new CKLS Model is proposed by introducing GARCH-type volatility with errors dis- tributed as the Standardised Standard Asymmetric Exponential Power Distribution (SSAEPD). The continuous time model in its general and restricted forms is estimated using the discrete- time form. In such a discrete world, this new model (i.e. CKLS-GARCH_SSAEPD) has following form:

$$\begin{aligned} r_t - r_{t-1} &= \beta_0 + \beta_1 r_{t-1} + r_{t-1}^c u_t \\ u_t &= \sigma_t \varepsilon_t \\ \sigma_t^2 &= a_0 + \sum_{j=1}^r a_j e_{t-j}^2 + \sum_{j=1}^s b_j \sigma_{t-j}^2 \\ a_0 &> 0, a_j \geq 0, j = 1, \dots, r, b_j \geq 0, j = 1, \dots, s. \\ 1 &\geq \sum_{j=1}^{max(r,s)} \left(a_j + b_j\right) \end{aligned} \tag{1}$$

Where $\theta = \{\beta_0, \beta_1, c, a_0, \{a_i\}^r, \{b_i\}^s, \alpha, p_1, p_2\}$ are parameters to be estimated. r_t is the interest rate data. β_0, β_1, c are the coefficient parameters in the regression model. The error term ε_t is distributed as the Standardised Standard Asymmetric Exponential Power Distribution (SSAEPD) proposed in Zhu and Zinde-Walsh [9]. σ_t is the conditional standard deviation, i.e., volatility. When $\{a_i = 0\}^r, \{b_i = 0\}^s$, the new model can be reduced to the CKLS-AEPD model of Li [7]. If $\{a_i = 0\}^r, \{b_i = 0\}^s, \alpha = 0.5$ and $p_1 = p_2 = 2$, the new model will be the classic CKLS model of Chan et al. [3].

2.2. Standardized Standard AEPD (SSAEPD)

In our new model, the error terms are assumed distributed as the Standardised Standard Asymmetric Exponential Power Distribution (SSAEPD) proposed by Zhu and Zinde-Walsh [9]. The probability density function (PDF) of the SSAEPD is:

$$f(\varepsilon_t \mid \theta) = \begin{cases} \delta\left(\frac{\alpha}{\alpha^*}\right) K(p_1) \exp\left(-\frac{1}{p_1}\left|\frac{w+\varepsilon_\varepsilon \delta}{2\alpha^*}\right|^{p_1}\right), & \text{if } \varepsilon_t \leq -\frac{w}{\delta} \\ \delta\left(\frac{1-\alpha}{1-\alpha^*}\right) K(p_2) \exp\left(-\frac{1}{p_2}\left|\frac{w+\varepsilon_t \delta}{2(1-\alpha^*)}\right|^{p_2}\right), & \text{if } \varepsilon_t > -\frac{w}{\delta} \end{cases} \tag{2}$$

Where

$$\alpha *= \frac{\alpha K(p_1)}{\alpha K(p_1) + (1-\alpha) K(p_2)}$$

$$K(p) = \frac{1}{2p^{1/p}\Gamma(1+1/p)}$$

$$\Gamma(x) = \int_0^\infty y^{x-1} e^{-y} dy$$

$$w = \frac{1}{B}\left[(1-\alpha)^2 \frac{p_2 \Gamma(2/p_2)}{\Gamma^2(1/p_2)} - \alpha^2 \frac{p_1 \Gamma(2/p_1)}{\Gamma^2(1/p_1)}\right]$$

$$\delta^2 = \frac{1}{B^2}\left\{(1-\alpha)^3 \frac{p_2^2 \Gamma(3/p_2)}{\Gamma^3(1/p_2)} + \alpha^3 \frac{p_1^2 \Gamma(3/p_1)}{\Gamma^3(1/p_1)}\right] - \left[(1-\alpha)^2 \frac{p_2 \Gamma(2/p_2)}{\Gamma^2(1/p_2)} - \alpha^2 \frac{p_1 \Gamma(2/p_1)}{\Gamma^2(1/p_1)}\right]^2\right\}$$

$$B = \alpha K(p_1) + (1-\alpha) K(p_2).$$

And $\mu \in R$, $\sigma > 0$, $p_1 > 0$, $p_2 > 0$, $\alpha \in (0, 1)$. p_1 and p_2 are the parameters which control the left tails and right tails, respectively. If $p_1 = p_2$, the SSAEPD reduces to a version of Skewed EPD (i.e., SEPD, see Theodossiou [10], and Komunjer [11]). If $p_1 = p_2 = 1$, it is the skewed Laplace distribution and if $p_1 = p_2 = 2$, it is the skewed Normal distribution. If $\alpha = 0.5$, the SSAEPD becomes symmetric EPD (see Bali and Wu's work [12]). For example, if $\alpha = 0.5$ and $p_1 = p_2 = 2$, the AEPD is the Normal distribution. If $\alpha = 0.5$ and $p_1 = p_2 = 1$, the AEPD is the Laplace distribution. The SSAEPD may capture the skewness, fat tails, asymmetry and jumps in the short rate.

2.3. Maximum Likelihood Estimation

We estimate the CKLS-GARCH_SSAEPD model with Maximum Likelihood Estimation (MLE). For simplicity, we define following notations $Y_t = r_t - r_{t-1}$. The likelihood function is:

$$f(Y_1, \dots, Y_T) = \prod_{t=1}^{T} r_{t-1}^{-c}\sigma_t^{-1} \begin{cases} \delta\left(\frac{\alpha}{\alpha*}\right)K(p_1)\exp\left(-\frac{1}{p_1}\left|\frac{w+\varepsilon_t\delta}{2\alpha^*}\right|^{p_1}\right), & \text{if } \varepsilon_t \le -\frac{w}{\delta} \\ \delta\left(\frac{1-\alpha}{1-\alpha^*}\right)K(p_2)\exp\left(-\frac{1}{p_2}\left|\frac{w+\varepsilon_t\delta}{2(1-\alpha^*)}\right|^{p_2}\right), & \text{if } \varepsilon_t > -\frac{w}{\delta} \end{cases} \tag{3}$$

Where

$$\varepsilon_t = r_{t-1}^{-c}\sigma_t^{-1}(Y_t - \beta_0 - \beta_1 r_{t-1})$$

$$\sigma_t^2 = a_0 + \sum_{i=1}^{r} a_i u_{t-i}^2 + \sum_{i=1}^{s} b_i \sigma_{t-i}^2$$

3. Simulation Analysis

In this section, we run following simulation for a CKLS-GARCH (1,1) _SSAEPD model. The data generation process is as follows.

Select a set of true parameter values for $\theta = \{\{\beta_0, \beta_1, c\}, \{\alpha, p_1, p_2\}, a_0, a_1, b_1\}$;

Generate SSAEPD random number series $\{\varepsilon_t\}_{t=1}T$;

Generate $\{\sigma_t^2\}$and $\{u_t\}$by following methods ($\sigma_0 = 1$, $u_0 = 0$).

$$\begin{gathered} \sigma_t^2 = a_0 + a_1\sigma_{t-1}^2\varepsilon_{t-1}^2 + b_1\sigma_{t-1}^2 \\ u_t = \varepsilon_t\sigma_t. \end{gathered} \tag{4}$$

Generate random number series $\{r_t\}$from Uniform(0,1) and get $\{Y_t\}_{t=1}^{T}$.

$$Y_t = \beta_0 + \beta_1 r_{t-1} + r_{t-1}^{c} u_t \tag{5}$$

After we have the simulated data, we can use the simulated data to estimate the parameters in the CKLS-GARCH_SSAEPD model. The simulation results are reported in Table 4. We find out the estimates are close to the true values even if we change the true parameters and re-run the simulation. Hence, we conclude the MatLab program is fine and can be applied to analyse empirical data.

Table 4. Simulation results.

	β_0	β_1	α_0	α_1	b_1	c	α	p_1	p_2
T	0.3	0.5	0.3	0.5	0.3	1	0.5	2	2
E	0.3001	0.4821	0.2388	0.4628	0.3785	1.0330	0.4973	2.0873	1.9858
T	0.3	0.5	0.2	0.5	0.3	1.5	0.5	1.5	2
E	0.3000	0.4922	0.1727	0.4744	0.3422	1.5352	0.5876	1.7613	1.9228
T	0.3	0.5	0.2	0.4	0.5	0.5	0.5	1.5	2
E	0.3029	0.4706	0.1422	0.3759	0.5535	0.5273	0.5870	1.9275	1.9393
T	0	0	1	0.4	0.5	1	0.5	2	2.5
E	-0.0003	-0.0339	0.8812	0.3776	0.5561	1.0051	0.4949	2.0930	2.5275
T	0.3	0.5	0.7	0.3	0.3	1	0.5	2	2
E	0.3003	0.4795	0.6755	0.2779	0.3975	1.0397	0.5024	2.0971	1.9595
T	0.3	0.4	0.6	0.5	0.4	1	0.4	2	2
E	0.3006	0.3687	0.5215	0.4652	0.4318	1.0356	0.3945	1.9244	1.8600

Notes: T= True values; E= Estimated values.

4. Empirical Analysis

4.1. Data

Chan et.al. analysed the one-month U.S. treasury bill yield data from 1964:06 to 1989:12 and showed that their model is better than the other interest rate models [3]. In this paper, we study the same data used in Chan et.al. (denoted as Sample A). We try to test the hypothesis whether our new model beat the CKLS model of Chan et.al.? Moreover, to check the robustness of CKLS model, we also apply the analysis to the sample from 1990:01 to 2013:12 (denoted as Sample B). All data are downloaded from the Center for Research in Security Prices (CRSP).

The descriptive statistics are calculated by EViews (Version 7.2) and listed in Table 5. For Sample A, the one-month yield r_t is slightly skewed to the right while the monthly change $r_t - r_{t-1}$ is skewed to the left. Both of them are highly leptokurtic. For Sample B, the monthly change $r_t - r_{t-1}$ is highly leptokurtic. Both of them are skewed to the right. All P-values of Jarque-Bera test are sero, which means both samples are not distributed as Normal under 5% significance level. It is necessary to change error terms from Normal to SSAEPD.

Table 5. Descriptive Statistics.

Sample	Variance	Mean	Max	Med.	Min	St.Dev	Ske.	Kur.	P
A	r_t	0.0056	0.0136	0.0051	0.0029	0.0022	1.1936	4.2956	0
	$r_t - r_{t-1}$	-0.0000	0.0025	0.0000	-0.0068	0.0008	-2.8111	25.6503	0
B	r_t	0.0220	0.0582	0.0225	0.0115	0.0157	0.0463	1.8539	0
	$r_t - r_{t-1}$	0.0001	0.0330	0.0001	-0.0020	0.0039	1.2372	22.9576	0

Notes: Medl=Median; Maxl=Maximum; Minl=Minimum; StlDel=Standard Deviation; Skel=Skewness; Kurl=Kurtosis; P=P-value of Jarque-Bera testl

Table 6 shows for both samples, the first six autocorrelations of the one-month yields and the monthly changes. The autocorrelations of interest rate decrease slowly while those of monthly changes, which are generally small, do not have a decline trend. That means that our samples are stationary.

Table 6. Autocorrelation Results.

Sample	Variance	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6
A	r_t	0.95	0.91	0.86	0.82	0.80	0.78
	$r_t - r_{t-1}$	-0.08	0.07	-0.12	-0.14	-0.03	-0.04
B	r_t	0.96	0.94	0.91	0.88	0.84	0.81
	$r_t - r_{t-1}$	-0.17	0.01	0.16	-0.07	-0.04	-0.05

4.2. Empirical Results

In this section, we estimate the new CKLS model with both samples. The estimation results are shown in Table 7. For the new model (i.e. Model 1), both estimates of skewness parameters α show slight skewness which is consistent with that observed in data. The estimates of both tail parameters p_1, p_2 (A: $\hat{p}_1 = 0.8346, \hat{p}_2 = 1.2843$, B: $\hat{p}_1 = 0.8321, \hat{p}_2 = 0.4428$) are much less than 2, which shows both tails are fatter than Normal. The constant terms in both mean equation (A: $\hat{\beta}_0 = 0.0001$, B: $\hat{\beta}_0 = -0.0018$) and volatility equation (A: $\hat{a}_0 = 0.3601$, B: $\hat{a}_0 = 0.1009$) are close to zero.

For comparison, we also estimate the CKLS_SSAEPD model (i.e. Model 2) by setting $p_1 = p_2 = 2$ and $\alpha = 0.5$ and the classic CKLS model (i.e. Model 3) by setting $a_1 = b_1 = 0$ and $a_0 = 1$. We can find that: 1) For both samples, the estimates of SSAEPD model are different from those of Normal models. 2) The estimates of c are similar in three models. 3) For Sample A, all estimates of c are bigger than 1, which means the conditional volatility of the process seems sensitive to the level of the short-term yield. However, they are smaller than 1 in Sample B. 4) For Sample A, the estimates of Model 3 are close to the estimated results from the study of Chan et.al.

4.3. Model Diagnostics

4.3.1 Test for Autocorrelation

In this subsection, we apply the Ljung-Box test to test residual autocorrelation. The null hypothesis of the Ljung-Box test is that all residual autocovariances are zero, that is, $H_0 : E(u_t u_{t-i}) = 0$ ($i = 1, 2\ldots$). Table 8 shows that the Ljung-Box test applied to the residual of the new model (i.e. CKLS-GARCH_SSAEPD) does not reject the null hypothesis of autocorrelations at lags 1 to 10 under 5% significance level. There exists no autocorrelation in error terms for our new model. Since the SSAEPD error has zero mean and unit variance, the SSAEPD error belongs to white noise process.

Table 8. Results of Ljung-Box Tests.

	Lag	1	2	3	4	5
	Q-statistic	0.1390	1.0210	2.3501	1.7040	4.3057
A	P-value	0.7093	0.6002	0.5030	0.7900	0.5063
	Lag	6	7	8	9	10
	Q-statistic	7.1870	9.0470	12.4962	15.0232	17.8187
	P-value	0.3039	0.2493	0.1304	0.0903	0.0581
	Lag	1	2	3	4	5
	Q-statistic	0.3684	0.6071	2.3970	3.3350	4.3756
B	P-value	0.5439	0.7382	0.4942	0.5034	0.4967
	Lag	6	7	8	9	10
	Q-statistic	5.9457	5.5638	9.4011	12.3728	15.8942
	P-value	0.4293	0.5915	0.3096	0.1931	0.1027

4.3.2 Test for Normality

In this subsection, we check the normality of the error terms in new CKLS model. We first apply Kolmogorov-Smirnov (KS) test with the hypotheses: H_0 : Residuals follow SSAEPD ($\hat{\alpha}, \hat{p}_1, \hat{p}_2$). Table 8 shows that both P-value of KS test are bigger than 0.05 significance level, which means the error terms do follow SSAEPD.

Then, from the graphes by method of "eye-rolling", we may also conclude the residuals of the new model fit SSAEPD better. That is, we compare the probability density function (PDF) of the estimated residuals $\hat{u}_t$ with that of SSAEPD ($\hat{\alpha}, \hat{p}_1, \hat{p}_2$). We find out these curves are very close to each other (see Figure 1). Then, we compare the PDF of the residuals with that of Normal (0, 1). We find out there are big differences between these curves (see Figure 2). Graph results show the residuals are more likely distributed as SSAEPD.

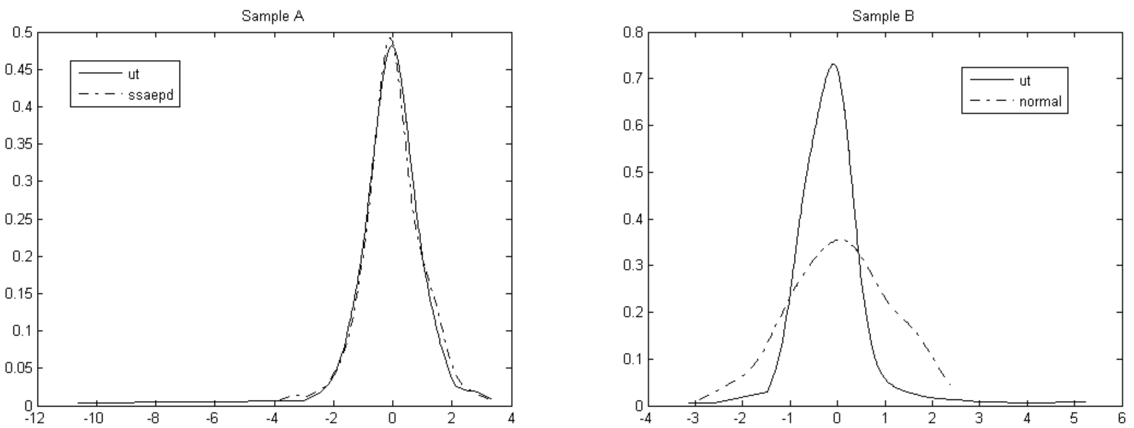

Figure 1. PDFs of $\hat{u}_t$ and SSAEPD ($\hat{\alpha}, \hat{p}_1, \hat{p}_2$). (a) PDFs of $\hat{u}_t$ and SSAEPD($\hat{\alpha}, \hat{p}_1, \hat{p}_2$), Sample A. (b) PDFs of $\hat{u}_t$ and SSAEPD($\hat{\alpha}, \hat{p}_1, \hat{p}_2$), Sample B

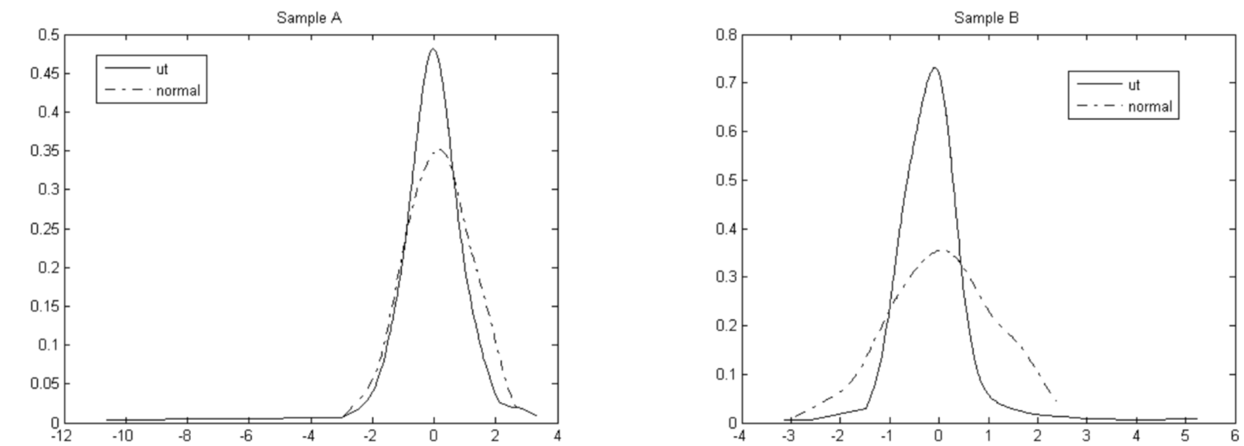

Figure 2. PDFs of uˆt and Normal (1,0). (a) PDFs of uˆt and Normal (1,0), Sample A. (b) PDFs of uˆt and Normal (1,0), Sample B.

4.3.3 Test for ARCH effect

Heteroskedasticity is often a concern for models based on data with daily frequency. If heteroskedasticity is found in the residuals, modelling them by GARCH models may be a useful way to avoid distortions in the mean parameters of the CKLS model. In this section, we first use the Engle-test to check the ARCH effects of the error terms et. The null hypothesis is H0 : There exists ARCH-effects in error terms. Table 9 shows that for Sample B, there seems to not exist the ARCH effects while for Sample A, the square of the residuals series does not reject the null hypothesis of ARCH effects at lags 5 to 10 under 5% significant level. Hence, it may be necessary to use GARCH-type volatility to describe the heteroskedasticity of interest rate.

Table 9. Results of Engle Test.

	Lag	1	2	3	4	5
	Q-statistic	15.1367	16.2235	14.2367	19.9974	20.5150
A	P-value	0.0001	0.0003	0.0026	0.0005	0.0010
	Lag	6	7	8	9	10
	Q-statistic	12.5374	13.208	13.0719	12.6417	12.4301
	P-value	0.0510	0.0672	0.1094	0.1795	0.2573
	Lag	1	2	3	4	5
	Q-statistic	9.5495	12.0646	17.3456	22.0046	21.6885
B	P-value	0.0020	0.0024	0.0006	0.0002	0.0006
	Lag	6	7	8	9	10
	Q-statistic	19.4982	13.9432	12.6946	14.1731	13.6984
	P-value	0.0034	0.0522	0.1228	0.1163	0.1872

4.3.4 Test for Parameter Restrictions

It shows there does not exist mean reversion feature in short-term interest rate [3]. In this section, we use Likelihood Ratio test (LR) 6 to test whether the mean equation is still alive in the new model and sample. First, we run the joint significance test for the regression coefficients. Both LR values of T4 are smaller than critical value 5.99, which means the slope coefficients of our model are not jointly statistically significant under 5% confidence level. Hence, it seems that there does not exist mean reversion feature in the short-term rate, which is same to the result in the work of Chan et al. [3]. Then, we run the individual significance test for the regression coefficients. In Table 10, we find both LR values of T6 are not statistically significant. Hence, we conclude that the relation between interest rate volatility and short-term field level may not be always highly sensitive during periods, which differs from Chan et.al. [3].

Table 10. Values of Likelihood Ratio Test Statistics (LR).

	T1	T2	T3	T4	T5	T6	T8	T9
A	0.60	0.40	564.6*	6.28*	90.97*	2.50	3.6	460.6*
B	21.02*	0.66	317.40*	6.26*	90.97*	0.94	38.72*	162.27*
χ²_.05	3.84	3.84	3.84	3.84	3.84	3.84	5.99	7.82

Notes: T1 means H0 : β0 = 0. T2 means H0 : β1 = 0. T3 means H0 : c = 0.
T4 means H0 : c = 0.5. T5 means H0 : c = 1.0. T6 means H0 : c = cˆ.
T7 means H0 : β0 = β1 = 0. T8 means H0 : β0 = β1 = c = 0.
* means the parameter restriction is statistically significant.
CV (5%) means the Critical Value under 5% Significance Level.

4.4. Model Comparisons

In this subsection, we compare our new model with the CKLS model of Chan et.al. [3] and other 6 models. The Akaike Information Criterion (AIC)7 is used as the model selection criterion. Table 11 lists the values of AIC. For both samples, the AIC values for our new model is smaller than others. Hence, we conclude that our new model is better than the CKLS model.

Table 11. Values of Akaike Information Criterion (AIC).

Models	M1	M2	M3	M4	M5	M6	M7	M8
A	-12.37	-12.10	-12.36	-12.11	-12.36	-12.18	-12.36	-12.36
B	-0.15	0.44	0.34	2.01	0.01	0.41	0.37	0.01

Notes: M1 means CKLS-GARCH SSAEPD. M2 means CKLS-GARCH Normal. M3 means CKLS SSAEPD. M4 means CKLS Normal. M5 means CKLS-GARCH SSAEPD with $\alpha = 0.5$. M6 means CKLA-GARCH SSAEPD with $p_1 = p_2 = 2$. M7 means CKLS-GARCH SSAEPD with $a_0 = 1$. M8 means CKLS-GARCH SSAEPD with $a_1 = b_1 = 0$.

5. Conclusion

In this paper, we introduce an enhanced version of the CKLS model, which we have named the CKLS-GARCH_SSAEPD model. This new formulation incorporates GARCH-type volatility and non-normal error terms to better address the complexities in modeling financial data. We applied this model to analyze one-month U.S. Treasury bill yield data using the Maximum Likelihood Estimation (MLE) method. Our approach began with a simulation analysis to validate the model's theoretical underpinnings. We then proceeded to empirically test the model on two distinct datasets: one covering the period from June 1964 to December 1989, and the other from January 1990 to December 2013. The goodness of fit for the model was assessed using the Kolmogorov-Smirnov test to analyze the residuals, while structural shifts were examined through the Likelihood Ratio test. Additionally, the Akaike Information Criterion (AIC) was employed for comparative model assessment. The empirical findings demonstrate that incorporating GARCH-type volatilities and an SSAEPD error distribution enhances the model's ability to capture the inherent skewness, kurtosis, and asymmetry observed in the data. The modified model outperformed the original CKLS model in terms of AIC values for both datasets, indicating a better fit. Interestingly, the analysis also revealed that the conditional volatility of the yield process shows variable sensitivity to the levels of short-term yields across different periods. Furthermore, our results suggest the absence of mean reversion in the short-term interest rate models for both periods studied.

Looking ahead, several avenues for further research present themselves. Future studies could explore the use of a Genetic Algorithm as an alternative to the interior-point method to potentially achieve a global optimization of the model parameters. Additionally, the robustness of our findings could be tested across different types of bonds to verify the model's applicability across broader market conditions. Lastly, the development of a multi-factor CKLS model based on the SSAEPD approach could provide deeper insights into the dynamics of interest rates and enhance the predictive power of the model.

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