Current Developments and Limitations for Pricing Options

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Abstract. Valuing options holds significant importance within the derivative market, serving as a cornerstone for investors, traders, and financial institutions. Over time, economists have dedicated efforts to develop models that accurately reflect the complexities of this market environment. The evolution of economic theories can be observed through the examination of key option pricing models, which have undergone refinements and enhancements over the decades. By exploring models such as the Black-Scholes model, the Merton model, and the Cox-Ross-Rubinstein model, it is possible to gain valuable insights into the progression of economic thought in this field. Each of these models offers unique perspectives on option pricing, illustrating various principles and methodologies employed to assess and value options. Moreover, understanding the extensions and limitations of these models provides crucial insights into their applicability and effectiveness in real-world scenarios. By conducting an in-depth examination of these models, the objective of this article is to offer a nuanced comprehension of how economic theories have developed and adjusted to the ever-changing conditions of the derivative market.

Keywords: Black-Scholes model, Merton model, Cox-Ross-Rubinstein model.

1. Introduction

Pricing options stands as a pivotal and intricate facet of financial operations, particularly within the derivative market, where options confer the ability to make a transaction. contracts before their maturity date, thereby enabling investors to prevent risks or engage in speculation regarding future price fluctuations. The increasing flexibility within the derivative market fosters plenty of opportunities and methodologies for profit acquisition, driving innovation and sophistication in financial strategies.

The journey of pricing options in the realm of financial mathematics traces back to the seminal work of Louis Bachelier and his Bachelier model, which was published in 1900. Bachelier, revered as the forefather of mathematical finance, spearheaded the exploration of stochastic processes and their application to modeling stock price movements. His pioneering efforts laid the groundwork for subsequent developments in option pricing theory. Despite decades of limited breakthroughs, the renowned Black-Scholes model emerged triumphantly onto the global stage in 1973, fundamentally reshaping the landscape of financial mathematics. This groundbreaking model provided a robust framework for pricing European options on non-dividend-paying stocks. During that crucial year, the Merton model became an extension of the Black-Scholes model. Created by Robert Merton, this extension enriched the Black-Scholes framework by incorporating the consideration of continuous dividends and other incomes into the option pricing process, thereby enhancing its applicability to a wider range of assets and market conditions. Furthermore, building upon the foundation laid by the Black-Scholes model, a mass of extensive models has successively emerged, each refining and expanding upon the original framework to address specific nuances and complexities of financial markets. Notably, in 1979, the Cox-Ross-Rubinstein model was introduced, providing a generalizable numerical method for the assessment of options' worth throughout the lifespan of the underlying financial asset.

2. Black-Scholes Model

In 1973, Fischer Black and Myron Scholes unveiled a groundbreaking formula capable of computing the price of European-style options and derivatives, particularly those lacking dividends during the option's lifetime. The model they proposed was based on several fundamental assumptions:

Efficient Markets: The model presupposes the efficiency of financial markets, positing the absence of arbitrage opportunities, thereby ensuring fair pricing across assets.

Continuous Trading: It assumes a seamless and continuous trading environment for the underlying asset, devoid of transaction costs or tax implications. The returns on the underlying asset follow a log-normal distribution.

Log-Normal Distribution: At the core of the model lies the premise that the returns of underlying asset follow a log-normal distribution.

Geometric Brownian Motion: The price movement is characterized by a geometric Brownian motion that is constantly drifting and volatile, creating a stochastic process for the dynamics of asset prices.

Constant Risk-Free Rate: The risk-free interest rate remains constant and known throughout the option's lifespan, enabling precise valuation and hedging strategies.

By these assumptions, the Black-Scholes formula simplifies the complex interplay of factors influencing option pricing. Specifically, for European put options, the formula computes the option price as

$$E = S_0 N(d_1) - Ke^{-r(T-t)} N(d_2) \tag{1}$$

with

$$d_1 = \frac{1}{\sigma\sqrt{T-t}} [\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)] \tag{2}$$

$$d_2 = d_1 - \sigma\sqrt{T-t} \tag{3}$$

where S_0 represents spot price, K denotes option’s strike price, r stands for risk-free interest rate, $T-t$ symbolizes duration of maturity, N is the distribution function of the standard normal distribution, and d_1 and d_2 are quantities computed utilizing the Black-Scholes formula. This formula serves as a cornerstone in the realm of mathematical finance, facilitating precise option pricing, risk management, and strategic decision-making in financial markets.

Grossman and Zhou developed a model that was constructed using the Black-Scholes model, but volatility becomes an endogenous variable that is influenced by both information about final payoffs and the desire for put protection. They generated a martingale approach that yielded better predictions than the Black-Scholes model under dynamic market situations [1]. The Black-Scholes model posits that the returns of underlying assets conform to a log-normal distribution. Other assumptions similar to the Black-Scholes model include using a negatively shifted log-normal distribution to value and hedge basket and spread options and utilizing a log-extended-skew-normal distribution to price Asian and basket options [2-3].

3. Merton Model

Soon afterwards, in 1974, Robert C. Merton published Merton model. The Merton model employs a geometric Brownian motion process to represent the fluctuation in the firm's asset value. It utilizes option pricing methodologies akin to those found in the Black-Scholes model to estimate the probability of the firm's assets declining below its debt commitments, potentially resulting in default. This assessment is encapsulated in two fundamental metrics: the Distance to Default (DD) and the Probability of Default (PD). The Distance to Default metric determines how far a firm's asset value is from its debt level, which gives insight into its financial stability. Meanwhile, the Probability of Default metric encapsulates the likelihood of a firm defaulting on its debt obligations, derived from the Distance to Default using standard statistical methodologies. The Distance to Default (DD) is defined by

$$DD = \frac{\ln\left(\frac{V}{D}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \tag{4}$$

and Probability to Default (PD) is

$$PD = N(-DD) \tag{5}$$

where V represents the market value of a company's assets, D denotes the market value of a company's debt, r is risk-free interest rate, T symbolizes duration of maturity, and σ is the volatility of the company's asset value. The relationship between a firm's asset value, liabilities, and equity can be leveraged to price equity options and inform option trading strategies.

Continuous research focuses on methods of predicting firms' assets and the probability of bankruptcy. In terms of forecasting bankruptcy, the hazard model is theoretically superior to static models [4]. Furthermore, researchers have improved the hazard model by adding non-financial factors, such as balance sheet ratios and market variables [5].

There is continuous research on the measurement of firms' defaults. In 1989, the KMV model emerged similarly to the Merton model, as an extension and application of the original Merton model. It incorporated additional features and enhancements to better assess and manage credit risk in practice. The KMV model expands upon the Merton model, offering a more comprehensive framework to estimate the probability of default (PD) and assess credit risk for a firm's debt securities. In 1994, the Leland model extended the Merton model by incorporating the presence of corporate taxes and the tax shield provided by debt interest payments, allowing for a more accurate estimation of the value of debt and PD. The expansion of the Leland model continued with the introduction of cash holding and dividend payment, which combined liquidity and solvency concerns of the firm [6]. Additionally, an illiquid secondary bond market was incorporated in the Leland-Toft model [7].

4. Cox-Ross-Rubinstein Model

The Cox-Ross-Rubinstein (CRR) model, introduced by John Cox, Stephen Ross, and Mark Rubinstein in 1979, is a popular binomial options pricing model extensively employed in financial markets because of its simplicity and effectiveness in valuing options. The foundation of the CRR model lies in the premise that the asset price undergoes periodic fluctuations, moving either upwards or downwards, with the anticipated return on the option matching the risk-free rate. The CRR model gives out the risk-neutral probability and option price at each node:

$$p = \frac{e^{r\Delta t} - d}{u - d} \tag{6}$$

$$C = e^{-r\Delta t}(p \cdot C_{up} + (1-p) \cdot C_{down}) \tag{7}$$

$$P = e^{-r\Delta t}(p \cdot P_{up} + (1-p) \cdot P_{down}) \tag{8}$$

where r is the risk-free interest rate, C_{up}, C_{down} are the prices of call options at the ascending and descending nodes, respectively, and P_{up}, P_{down} are the put option prices at up and down nodes respectively. Usually, in the case of European options, the prices of options at the terminal nodes correspond directly to the option payoffs. However, for American options, the prices at the terminal nodes are determined as the higher value between the option payoffs and the intrinsic values of the options.

Broadie and Detemple and Heston and Zhou suggested replacing the value of option before maturity, referred as continuation value, by Black and Scholes formula and adding Richardson extrapolation in the CRR Model which is called the Binomial Black and Scholes Method with Richardson Extrapolation, giving a simple way to calculate today's spreadsheets [8]. This model makes the binomial prices converges to Black-Scholes price uniformly and Richardson extrapolation derives the solutions with high accuracy [9]. Cox-Ross-Rubinstein model is mostly used to price European option price accurately. Researcher works on expand Cox-Ross-Rubinstein model on more option types: Hussian and Khan expand CRR model on evaluating compound options [10].

5. Comparison of These Models

Models are used to simulate realistic situation but they also are limited in reflecting every aspect of real world. Black-Scholes model, Merton model and Cox-Ross-Rubinstein Model considering constant volatility of assets price which is equivalently to consider the price of assets fluctuate according to Brownian motion and no dividend paid during the process is unrealistic. Volatility is crucial to value the options. M. Gultekin and B. Gultekin tested data on stock returns, equity returns, and taxes paid in 17 capital markets over approximately 20 years. A seasonal pattern in stock returns was observed in most countries, with a significant mean return observed at the turn of the tax year, typically in January, according to their findings [11]. This observation suggests that financial data exhibit a propensity to be discontinuous. Macbeth and Merville state that the unstable variance of stock prices in a stochastic process could cause Black-Scholes model prices of in-the-money options to be greater than market prices and out-of-the-money options to be less than market prices. This inconsistency reflects the reality that the volatility of asset prices may vary over time and differ across different options contracts [12]. The implicit volatility, originating from the Black-Scholes model, characterizes the volatility level of the underlying asset's price. However, its accuracy might be compromised in instances of heightened volatility within the financial market. The high volatility also causes limitations in Merton model. Although the measure of distance-to-default (DD) serves as a dependable metric for evaluating a company's default risk in accordance with the Merton model, if the asset experiences large jumps or stochastic volatility, the Merton model's distance-to-default measurement has lower precision in assessing a company's performance. When the market is not close to efficient market, applying Merton's model would lead to significant inaccuracy. Any unadvisable decision made by institution will lead to great loss probably leading to bankruptcy [13]. By employing an adjusted distance-to-default measure that estimates a stochastic asset volatility specification, firms with stochastic volatility can have better performance assessment [14]. For CRR model, facing stochastic volatility would cause problem. According to the CRR model, the price of the underlying asset is limited to a fixed percentage increase or decrease at every time step. In high-volatility environments, where price movements are larger and more erratic, this fixed-percentage assumption may lead to inaccurate price estimates. The model may not capture the full extent of price fluctuations, resulting in option prices that deviate from market prices.

Furthermore, the 2008 financial crisis has caused banks to no longer be considered risk-free for corporate transactions. Using Black-Scholes model to value the price of derivative becomes inaccurate during that time. The factors considered in Black-Scholes model can be reasonably extended. Burgard and Kjaer, Hull and White, and Hull and White points out considering factors of CVA, FVA, and DVA is beneficial [15-17]. Similarly, to the collapse of Black-Scholes model in 2008 financial crisis, it is reasonable to add CVA, FVA and DVA in Cox-Ross-Rubinstein model [18].

No option pricing model exists that is entirely lack of limitations. Each model requires simplifying assumptions regarding market dynamics and investor behavior, which may not consistent with real-world practices. Moreover, financial markets, characterized by their complexity and continual evolution, present an enormous challenge in designing a singular model capable of inclusively capturing all factors of option pricing across diverse market conditions. Consequently, various option pricing models have been formulated to target specific limitations and more accurately reflect real-world market dynamics. For instance, certain models endeavor to integrate features such as stochastic volatility, asset price jumps, or market frictions to enhance precision.

Despite the imperfections inherent in every model, ongoing research and advancement in financial mathematics persistently refine existing models and foster the development of novel approaches to option pricing.

6. Conclusion

This article provides an in-depth exploration of three pivotal option pricing models: the Black-Scholes model, the Merton model, and the Cox-Ross-Rubinstein model, with a keen focus on illustrating their structures and inherent limitations. Specifically, the article highlights the notable constraints encountered within the Black-Scholes model, Merton model, and Cox-Ross-Rubinstein Model particularly their challenges in accurately predicting outcomes in highly volatile financial market environments. The transformative impact of the 2008 financial crisis emerges as a pivotal juncture, catalyzing a shift in the realm of option pricing models. In response to the shortcomings exposed by the crisis, both the Black-Scholes and Cox-Ross-Rubinstein models underwent significant enhancements, incorporating additional factors to enhance their predictive capabilities and robustness. By delving into these models and their adaptations, this article offers a complete understanding of the dynamic evolution and ongoing refinement of option pricing methodologies in response to real-world challenges and market complexities.

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