# Application of Capital Asset Pricing Model in Finance

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**Abstract.** In the financial sector, Mathematics is indispensable everywhere. To be precise, Portfolio Optimization and Risk Management are two common methods under investment. The article introduces Portfolio Optimization and Risk Management to explain how mathematics using in Finance. In Portfolio Optimization plate, this article starts with the basic theories, which include Markowitz portfolio theory and combine with the celebrated Capital Asset Pricing Model (CAPM). In Risk management, the article specifically describes how do calculus used in Value-at-Risk (VaR) and conditional VaR. This article uses specific data and function to explain how math using in these sectors. CAPM model makes it possible for investors to assess and choose among competing financial based on absolute risk as opposed to overall risk. The primary benefits of CAPM are its clarity and simplicity. It breaks down the cost of any risk security into three components: risk price, risk calculation unit, and risk-free rate of return. These three components are then naturally combined.

**Keywords:** Value-at-Risk; Capital Asset Pricing Model; Finance.

## 1. Introduction

In contemporary society, mathematics is used in everywhere, especially in finance sector. This article provides two most common computational requires mathematics computations, which are Portfolio Optimization and Risk Management. Markowitz Portfolio theory is the theoretical basis of portfolio optimization. In 1952, Markowitz, an American economist, first proposed Portfolio theory. In Markowitz portfolio theory, Capital Assert Pricing Model (CAPM) is used [1]. This model can be used to figure out the connection between investment risk and expected return. The expected return of an assert is thought to have a positive correlation with the beta value, a scale used to assess an assert's risk. Market equilibrium is explained by the CAPM, which is predicated on the idea that the theory of investment management is applicative for all investors. Furthermore, it illustrates the underlying link between the expected return and expected risk of an asset.

It ought to be mentioned as a theory that clarifies how the Risky's equilibrium price claims that the one index simplifies the portfolio selection calculation process, marking Markowitz's theory of choosing a portfolio a major step towards real-word application. It also transforms securities theory from prior qualitative analysis to quantitative analysis and displays a significant impact on the theoretical investigation and day-to-day operations pf securities investment, as well as the evolution of financial theory and practice as a whole. As a result, it serves as the theoretical foundation for modern finance. Calculus is mostly used in risk management to determine the difference between risk indicators, including Value-at-Risk (VaR) and conditional VaR (CVaR) [2]. Compared to the former, CVaR is more enticing. It is worthwhile to note that the topic of choosing the best VaR and CVaR portfolios is examined in this article.

This paper demonstrated that majority of ill-posed derivatives portfolio minimization issues for VaR and CVaR. This paper suggests adding cost to the list of preference criteria for the CVaR optimization problem. Demonstrate that, if appropriate cost is applied, an ideal CVaR derivative investment portfolio may be calculated with a far smaller number of instruments and equivalent CVaR and VaR. Additionally, these articles describe how math is used in risk management and portfolio optimization using a few mathematical formulas.

# 2. Portfolio Optimization

## 2.1. Definition of Portfolio Optimization

The portfolio optimization is helpful to recombine asserts and distribute risks in accordance with the specified target return and risk tolerance. It represents both the desires of investors and the limitations imposed upon them. An optimal portfolio is one that precisely strikes a balance between returns and risk. An optimal portfolio is one that strikes a balance between securities with the lowest risk at a given return and those having the highest potential returns at a manageable risk. Calculus is used to solve a portfolio's risk and return functions in order to determine the ideal portfolio [3]. The theoretical foundation of portfolio optimization is Markowitz portfolio theory. The CAPM is utilized in Markowitz portfolio theory. The CAPM shows how the risk-free return is increased by the computed risk premium. Equals the anticipated rate of return on investment as determined by the security's beta. It is believed that riskier asserts must yield a higher expected return in order to persuade investors to hold them. A measure of this risk-return relationship is provided by the CAPM. The CAPM has been extensively utilized in the financial sector, including capital planning, portfolio creation, and project appraisals inside firms. This is attributed to its easily understood mathematical representation of how risk and reward are related. Policymakers also utilize the CAPM to evaluate how altering policy will impact risk.

## 2.2. Capital Asset Pricing Model

It is clear that CAPM model is a notable development in the world of finance that focused on the distinct relationship that occurs between a single asset $(i)$ and the market. First, the author presents the security's beta, which indicates the possibility that its return moves in lockstep with the market's return [4]

$$\beta_i = \frac{Cov\ (R_{i,}\ R_M\ )}{\sigma^2{}_M} = \rho_{iM}\frac{\sigma_i}{\sigma_M}. \tag{1}$$

For a given security, this passage can derive additional crucial details about the security by estimating $\beta$. This passage can, for instance, have a lower bound in some cases. How does the $\beta$ coefficient assist people in determining the valuation of a security? To address this issue, the following theorem is helpful.

Theorem 1. If $\mu$ and $e$ are linearly independent and the covariance matrix $V$ is positive definite, then $\mu_i - r = \beta_i(\mu_M - r)$.

*Proof*: the next step is to rewrite the factor as

$$\beta_i = \frac{\mathbf{e}_\mathrm{i}^\mathrm{T}\mathrm{V}\mathbf{w}_\mathrm{M}}{\mathbf{w}_\mathrm{M}^\mathrm{T}\mathrm{V}\mathbf{w}_\mathrm{M}} = \frac{\mu_\mathrm{i} - \mathrm{r}}{\mu_\mathrm{M} - \mathrm{r}}. \tag{2}$$

The author shall calculate the covariance first, which is

$$Cov(R_i, R_M) = Cov(R_i, \boldsymbol{w}_M^T) = Cov\left(R_i, \sum_{l=1}^{N} w_l^M R_l\right) = \sum_{l=1}^{N} w_l^M\, Cov(R_i, R_l) = e_i^T V_{W_M} \tag{3}$$

So, it is direct to find that $\beta_i = \frac{e_i^T V w_M}{w_M^T V w_M}$. However, since $w_M = \frac{V^{-1}(\mu_M - r_e)}{B - A_r}$, then $e_i^T V w_M = e_i^T \frac{\mu_M - re}{B - Ar} = \frac{\mu_i - r}{B - Ar}$ and $w_M^T V w_M = w_M^T \frac{\mu_M - re}{B - Ar} = \frac{\mu_M - r}{B - Ar}$. So, at last one can find that $\beta_i = \frac{\mu_i - r}{\mu_M - r}$.

Here is an example to apply the above theory. Suppose that $r = 2\%$ and $\mathbb{E}[R_A] = 20\%$, then one can know that $\mathbb{E}[R_A] = 0.02 + 1.5 \times (0.2 - 0.02) = 0.29$ and $\mathbb{E}[R_B] = 0.02 + 1.4 \times (0.2 - 0.02)$=0.29. It is crucial to utilize the appropriate discount rate when calculating net present values. Here, the author is attempting to compare the decision-marking to the anticipated market return. AS a result, people already have

$$NPV_B = -10{,}000 + 9000\left(\frac{1}{1.272^2} + \frac{1}{1.272^5}\right) \approx -412 < 0 \tag{4}$$

This proves that both projects are worthless and should be abandoned. Keep in mind that a varied predicted market return and beta could change the entire picture. The following interpretations are available. The line's slope indicates the degree to which a given security is associated with the market. The risk premium is $\beta(\mu_M - r)$. Whether the market and the security move more likely in the same direction or the opposing way is indicated by the sign of $\beta$.

Why is the pricing model known as the CAPM model? It turns out that this model can be helpful in pricing an asset's current worth when combined with knowledge about $\beta$. Think about the asset's return over a given length of time $R = \frac{S_T - S_0}{S_0}$. With anticipated values $\mathbb{E}[R] = \frac{\mathbb{E}[S_T] - S_0}{S_0}$. Therefore, it is inferred that $S_0 = \frac{\mathbb{E}[S_T]}{1+\mathbb{E}[R]}$. Through the CAPM, one can reprocess

$$S_0 = \frac{\mathbb{E}[S_T]}{1 + r + \beta(\mu_M - r)} \tag{5}$$

Remarks. It is imperative to utilize the distribution of $S_T$ for the valuation of an asset within this model; Furthermore, the accurate of $\beta$ emerges as a critical component in. the pricing mechanism. This approach ensures the integration of market inputs into the asset valuation process. The formula for calculating the hurdle rate is $h = \mathbb{E}[R_{project}] = r + \beta_{project}(\mathbb{E}[R_M] - r)$. For some data, the lowest allowable return on investment for investors is the project or venture's hurdle rate. It is a very helpful component for investors to consider when marking decisions.

## 3. Applications in Risk Management

The process of identifying, evaluating, and controlling hazards in financial market is known as risk management. Numerous aspects, inencompassing market, operational, and credit risk, and so on, are involved in risk management. Calculus is mostly used in risk management to determine how different risk indicators, like VaR andCVaR, are from one another. It is common practice to employ them at risk assessments to VaR. The author proposes to incorporate cost as an extra preference criterion for the CVaR optimization problem. It is demonstrated that, if appropriate cost is applied, an ideal CVaR derivative investment portfolio may be calculated with a far smaller number of instruments and equivalent CVaR and VaR [5].

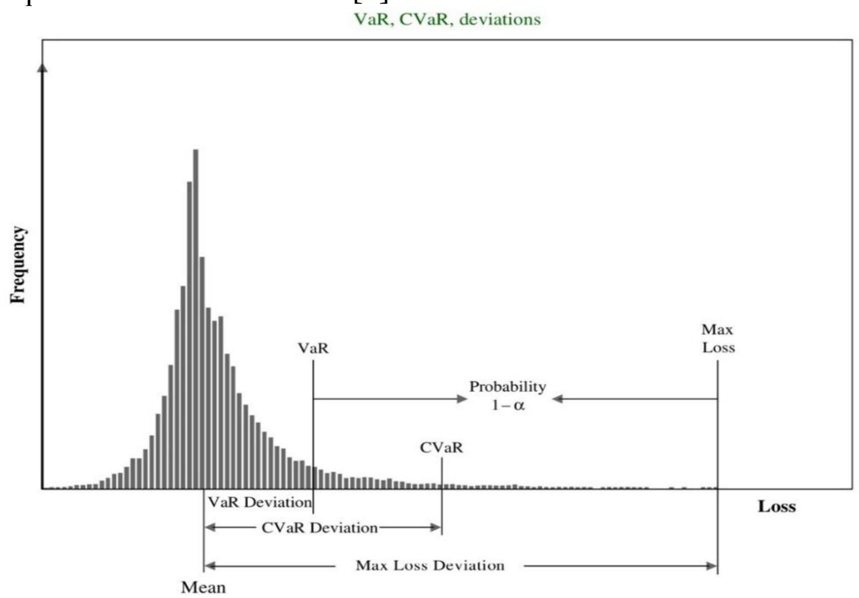


**Fig. 1** The following risk functions are displayed graphically: VaR, VaR Deviation, CVaR, CVaR Deviation, Max Loss, and Max Loss Deviation [5].

The risk functions on VaR, VaR Deviation, CVaR, and CVaR Deviation are displayed in Fig. 1. A computational technique based on a smoothing methodology is constructed to efficiently solve a simulation-based CVaR optimization problem. Let $x$ be a random variable and use the cumulative distribution function. Gain or loss could in indicated using the formula $F(z) = P\{X \leq z\}$. The signal functions in the VaR and CVaR definitions changes as a result of the tutorial' use of $X$ to symbolize loss

### 3.1. Definition of VaR

The VaR of X with confidence level $\propto \in [0,1]$ is Va$R_{\propto}$(X)=min$\{z|\, Fx(z) \geq \alpha\}$. On the one hand, value-at-risk is widely utilized in several technical domains that involve uncertainly, including but not limited to military, nuclear, material, aerospace, finance, and so on. VaR deviation, for instance, is used to calculate the width of a portfolio's daily loss distribution in financial regulations. For a Gaussian distributed random variable, VaR is linear to its standard deviation. If $Fx(z)$ is the cumulative distribution function of the form $X \sim N(\mu, \sigma)^2$ and $Fx(z)$ is the cumulative distribution function of X ,then Va$R_{\propto}$(X)=$F_x^{-1}(\alpha) = \mu + k(\alpha)\sigma$, where $k(\alpha) = \sqrt{2}\mathrm{erf}^{-1}(2\alpha - 1)$ and $erf(z) = (2 / \sqrt{\pi}) \int_0^z e^{-t^2}\,\mathrm{dt}$. VaR′s mathematical characteristics balance off its simplicity and intuitiveness. Va$R_{\alpha}$(x) is a nonconvex, a jump function for discrete distributions as a function of the confidence level.

### 3.2. Definition of CVaR

An additional percentile risk metric is called CVaR. The confidence level $\alpha \in [0,1]$. For $X$'s CVaR is the mean of the generalized a-tail distribution

$$CVaR_{\alpha}(X) = \int_{-\infty}^{\infty} z\, dF_X^{\propto}(z) \tag{6}$$

where

$$F_X^{\propto}(z) = \begin{cases} 0, \text{ when } z < \mathrm{Va}\, R_{\alpha}(X), \\ \dfrac{F_X(z) - \alpha}{1 - \alpha}, \text{when } z \geq \mathrm{Va}\, R_{\alpha}(X\,). \end{cases} \tag{7}$$

Unlike what is commonly believed, in the most cases, $CVaR_{\alpha}(X)$ does not equal an average of a outcomes that are higher that than $VaR_{\alpha}(X)$. A probability atom might need to be divided for generic distributions. For instance, a small number of possible outcomes may be averaged to determine C$VaR$ when the distribution is modeled by scenarios. This article also presents alternate definitions of $CVaR$ in order to provide a more thorough explanation of the concept. Let $CVaR_{\alpha}^{+}(X)$ ,also known as "upper CVaR", be the contingent anticipation of $X > VaR_{\alpha}(X)$.

An alternate definition of $CVaR_{\alpha}(X)$ would be the weighted mean of $CVaR_{\alpha}^{+}(X)$ and $VaR_{\alpha}(X)$.If the $Fx(VaR_{\alpha}(X)) < 1$, then there is a higher likelihood of a loss than $VaR_{\alpha}(X)$ ,then

$$CVaR_{\alpha}(X) = \lambda_{\alpha}(X)VaR_{\alpha}(X) + (1 - \lambda_{\alpha}(X))CVaR_{\alpha}^{+}(X) \tag{8}$$

where

$$\lambda_{\alpha}(X) = \frac{F_X(VaR_{\alpha}(X)) - \alpha}{1 - \alpha}. \tag{9}$$

Conversely, if $Fx(VaR_{\alpha}(X)) = 1$, meaning that $VaR_{\alpha}(X)$ is the maximum possible loss, then $\mathrm{CVaR}_{\alpha}(\mathrm{x}) = \mathrm{VaR}_{\alpha}(\mathrm{x})$.

The definition of CVaR shown in Eq. (4) shows that it is not what's meant to be seen as conditional. If continuous distributions, the function of $\mathrm{CVaR}_{\alpha}^{-}(\mathrm{X}) = E[X \mid X \geq VaR_{\alpha}(X)]$, also referred to as "lower CVaR" like $\mathrm{CVaR}_{\alpha}(\mathrm{x})$. In general distributions, however, it is discontinuous and is not convex. One significant innovation is the way that $\mathrm{CVaR}_\alpha$ is constructed as a weighted average of $\mathrm{VaR}_\alpha$ and $\mathrm{CVaR}_\propto^+(X)$ .Even though CVaR is very desirable function, VaR and $\mathrm{CVaR}_\propto^+(X)$ do not behave well as measures of risk for generic loss distributions (both are continuous functions). It is jointly convex in $(X,\alpha)$ and continuous with regard to $\alpha$. One peculiarity in the definition of CVaR is the ability to split the VaR atom. This is a confidence level $\alpha$ interval using the same VaR, if $F_X(x)$ has a vertical discontinuity gap. $\alpha^- = F_X\big(VaR_\alpha^-(X)\big)$ and $\alpha^+ = VaR_\alpha(X))$, where $F_X\big(VaR_\alpha^-(X)\big) = P\{X < VaR_\alpha(X)\}$. The atom $\mathrm{VaR}_\alpha(\mathrm{X})$ with total probability $\alpha^+ - \alpha^-$ is divided into two halves with probabilities When $\alpha^+ - \alpha^-$ and $\alpha - \alpha^-$ by the confidence level $\alpha$ when $F_X\big(VaR_\alpha^-(X)\big) < \alpha < F_X VaR_\alpha(X)) < 1$. This separation is highlighted in Eq. (4).

The following examples further clarify the notion of CVaR. Assume that there are six equally likely situations, each with a loss of $f_1 \dots f_6$ [6]. In Fig. 2, if $\alpha = 1$, then $\alpha$ does not spilt any probability atom in this instance. Consequently, $VaR_\alpha(X) < CVaR_\alpha^-(X) < \mathrm{CVaR}_\alpha(\mathrm{X}) = CVaR_\alpha^+(X), \lambda_\alpha(X) = (F_X VaR_\alpha(X)) - \alpha) \,/\, (1-\alpha) = 0$ and $\mathrm{CVaR}_\alpha(\mathrm{X}) = CVaR_\alpha^+(X) = \frac{1}{2}\mathrm{f}_5 + \frac{1}{2}\mathrm{f}_6$, where $f_5$ and $f_6$ denote the fifth and sixth losses respectively. Let's now look at $\alpha = \frac{2}{3}$ .Here, $\alpha$ does divide the $\mathrm{VaR}_\alpha(\mathrm{X})$ atom, $\mathrm{CVaR}_\alpha(\mathrm{X})$ is given by, and $\lambda_\alpha(X) = (F_X VaR_\alpha(X)) - \alpha) \,/\, (1-\alpha) > 0$. In this regards, it is found that $\mathrm{CVaR}_\alpha(\mathrm{X}) = \frac{1}{5}\mathrm{VaR}_\alpha(\mathrm{X})\mathrm{A} + \frac{4}{5}\mathrm{CVaR}_\alpha^+(\mathrm{X}) = \frac{1}{5}\mathrm{f}_4 + \frac{2}{5}\mathrm{f}_5 + \frac{2}{5}\mathrm{f}_6$.

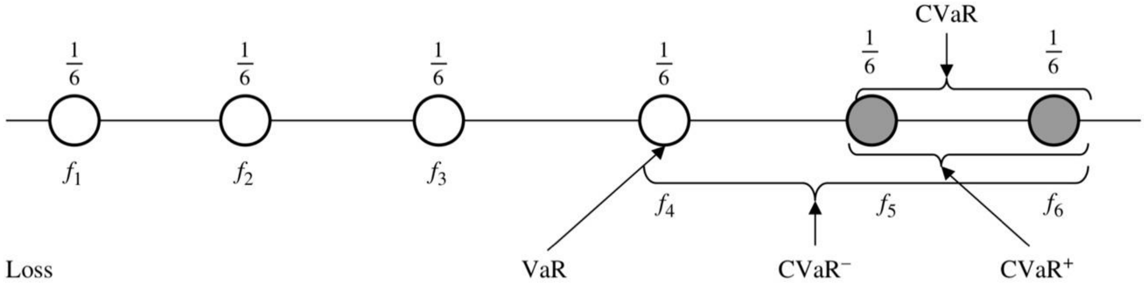


**Fig. 2** Example for CVaR computation: in the case where a does not cause atom splitting [6].

In the final scenario, $\alpha = \frac{7}{8}$ breaks the final atom and the author examines four equally likely outcomes. Here, $VaR_\alpha(X) = CVaR_\alpha^-(X)$, and there is no definition for $\mathrm{CVaR}_\alpha(\mathrm{X})$, upper CVaR, $\mathrm{CVaR}_\alpha^+(\mathrm{X})$ at this point. $\lambda_\alpha(\mathrm{X}) = (\mathrm{F_X VaR_\alpha(X)}) - \alpha) \,/\, (1-\alpha) > 0$ and $\mathrm{CVaR}_\alpha(\mathrm{X}) = VaR(X) = f_4$. Through the lower CVaR and upper CVaR, for discete distributions, the CVaR function is defined by the PSG package in a manner analogous to Eq. (4). Assume that $\mathrm{VaR}_\alpha(\mathrm{X})$ atom with a total probability of $\alpha^+ - \alpha^-$ is divided into two parts with probabilities of $\alpha^+ - \alpha^-$ and $\alpha - \alpha^-$ by the confidence level $\alpha$. Next,

$$\mathrm{CVaR}_\alpha(\mathrm{X}) = \frac{\alpha^+ - \alpha}{\alpha^+ - \alpha^-}\frac{1-\alpha^-}{1-\alpha}\mathrm{CVaR}_\alpha^-(\mathrm{X}) \tag{10}$$

Where $\mathrm{CVaR}_\alpha^-(\mathrm{X}) = \mathrm{E}[\,\mathrm{X} \mid \mathrm{X} \geq \mathrm{VaR}_\alpha(\mathrm{X})]$ and $\mathrm{CVaR}_\alpha^+(\mathrm{X}) = \mathrm{E}[\mathrm{X} \mid \mathrm{X} > \mathrm{VaR}_\alpha(\mathrm{X})]$.

Taking a different tack, Pflug proposed defining CVaR using an issue with optimization that he took from Rockafeller and Uryasev

$$\mathrm{CVaR}_\alpha(\mathrm{X}) = {}^{min}{}_C\{C + \frac{1}{1-\alpha}E[X-C]^+. \tag{11}$$

Acerbi provided an additional analogous representation of CVaR demonstrating that CVaR is the sane as the "anticipated deficit" as specified by $\mathrm{CVaR}_\alpha(\mathrm{X}) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_\beta(\mathrm{X})\mathrm{d}\beta$. It is also of importance to note that $CVaR_\alpha(X) = E[X \mid X \geq VaR_\alpha(X)] = \mu + k_1(\alpha)\sigma$.

### 3.3. Example

Assume a portfolio manager oversees a single asset in the portfolio. The return from the assertion has a Gaussian distribution with an annual average return of 10% and an annual typical deviation of 30%. The portfolio is currently worth $100 million. This article wishes to respond to a number of straight forward inquiries regarding the portfolio value distribution at year's end. How is the portfolio value distributed after the conclusion of the year? How likely is it that the worth will wind up being less than $80 million or that more than $20 million will have been lost? What is the biggest loss that can occur at the conclusion of the year with a 1% probability? The VaR at 1% is this.

First, the author loads the statistics package from Mathematica: Requirements ["Master of Statistics'"] and Needs ["Statistics' MultiDescriptiveStatistics'"]. Next, finding out the distribution of the year-end portfolio value is the first thing one must. There is a 15.9% chance that the portfolio worth the year's conclusion will be lower than $80. The portfolio at year's end will be less than 40.2096 with a probability of 1%; hence, the distribution's variance is 100-40.2096 =59.7904. Formalizing this would include developing a VaR function with the distribution's mean, deviation, and CVaR level as parameters.

## 4. Conclusion

To sum up, this paper shows Mathematics is everywhere in Finance sector. The CAPM is based on Modern Assert Allocation Theory (MPT) by Markowitz. This model describes the relationship between an assert's return rate, its risk factor $\beta$, and systematic risk using a straightforward mathematical formula. Furthermore, CVaR has a very major flaw in that it doesn't give any indication of the potential magnitude of losses belong the threshold amount that this metric indicates. It cannot distinguish in between circumstances where losses are more severe but might be considered merely somewhat worse and situations where losses are possibly terrible. Rather than exhibiting the conservatism that should govern risk management, it rather shows a bias toward optimism and provides merely a lower bound for loose in the tails of the loss distribution. Another measure called conditional value-at-risk, or CVaR, evaluates possible tail losses. In numerous aspects, CVaR is a superior tool for optimization modeling. However, the most significant expression for CVaR in terms of applications is a spectacular mitigation formula. This can be easily implemented in optimization problems regarding $x \in X$ that aim to control or constrain risk. This allows for significant shortcuts to be taken while maintaining important properties of the issue such as convexity. It is believed that in the future, mathematics will be used more in the financial industry to assist individuals make fewer mistakes and enjoy easier lives.

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