# Kelly Criterion and its Application

Xinyi Kong
Capital Normal University, Beijing 10015, China

**Abstract.** Kelly Criterion is a betting strategy that maximizes the long-term growth rate of principal in independent repeated bets. Kelly formula is beneficial to fund management and portfolio investment, and its application in investment is particularly prominent. In this paper, we take Kelly's criterion as the center, give the proof of Kelly's criterion with different methods and angles, and generalize it to a more general form. At the same time, this paper also illustrates the rationality of Kelly's criterion through examples and data simulation.

**Keywords:** Kelly Criterion; Mathematical Expectation; Optimization.

## 1. Introduction

Kelly formula is a betting strategy that maximizes the long-term growth rate of principal in an independent repeated bet with positive expected net income. In the previous research on Kelly formula, the focus was mostly on the deformation of Kelly formula, lacking the experimental simulation of the formula and the explanation of the application of Kelly formula. In this paper, the application of Kelly formula and the rationality of Kelly formula through experimental simulation are supplemented. The significance of the research is to better explain the advantages and disadvantages of Kelly formula in practical application. In this paper, we introduce the example of dice as a way to maximize the long-term growth rate of principal, and then verify the rationality of Kelly formula through data simulation, and generalize Kelly formula to a more general form. Finally, this paper introduces the application of Kelly formula in the housing market and the stock market, and then explains the advantages and disadvantages of Kelly formula.

## 2. The Kelly Formula

Let's start with a small game and explain a way to maximize the long-term growth rate of principal. There is a game of dice. You can bet k dollars at will. If you throw odd numbers 1, 3, 5, you will win 3 times of k dollars. If you throw even numbers 2, 4, 6, you will lose k dollars. Suppose you have 100 dollars, how much do you want to bet? Suppose that people are risk aversion, and the utility function is a logarithmic utility function, that is, U=lg (w), w is the finance after gambling, and the utility in different states can be directly added. U=1/2lg (w1)+1/2lg (w2). Next, we will solve this problem under this assumption. In order to better illustrate the game rules, we use tables as auxiliary explanations.

**Table 1.** The rules of the game are explained (1)

| | |
|---|---|
| Odd number | ×3 |
| Even number | ×0 |

Obviously, the probability of placing odd and even points is ½, if you bet all the principal, that means you have ½ chance to get 300 dollars, but it is also likely to lose all. So investing in all money is not a good strategy. Let think another choice, if you invest too little capital, that means even if you win the game, the income will be increased at low speed. Therefore, we need analyze the mathematical expectation of logarithmic principal after a game, Suppose that people are risk aversion, and the utility function is a logarithmic utility function, that is, U=lg (w), w is the finance after gambling, and the utility in different states can be directly added and let set a bet of x% principal, than the Logarithmic mathematical expectation is:

$$\frac{1}{2}\lg 100(1-x+3x)+\frac{1}{2}\lg 100(1-x)$$
$$=\frac{1}{2}\lg 10^4(1+2x)(1-x)$$

In order to maximize the benefits, we should maximize the mathematical expectations that means the above expressions. Then calculate the maximum value of the derivative of the function.

Solution:

$$f(x)=(1+2x)(1-x)$$

$$f'(x)=-4x+1=0$$
$$,\Rightarrow x=\frac{1}{4}$$

Therefore, 25% investment is the most profitable. this example illustrates one way to maximize the long-term growth rate of principal. Let us watch another game, There is a game of rolling dice to guess odd and even numbers. You can bet k dollars at your will. If you guess the odd numbers 1, 3 and 5, you will win 4 times of k dollars. If you guess even numbers 2, 4 and 6, you will win 3 times of k dollars. It is stipulated that you can bet odd and even numbers at the same time. If there is 100 dollars, how do you choose to bet? In order to better illustrate the game rules, we use charts as auxiliary explanations.

**Table 2.** The rules of the game are explained (2)

| | |
|---|---|
| Odd numbers | ×3 |
| Even numbers | ×4 |

If 100 dollars is all wagered in even numbers, there are ½ get the chance to get 400 dollars, although the amount is large, there are also ½ opportunities are lost. A similar conclusion can be obtained if 100 dollars is all bet on odd numbers. Guess that it is more profitable to bet on odd and even numbers at the same time. Therefore, we need analyze the mathematical expectation of logarithmic principal after a game. Let's bet an even number with x% of the principal and an odd number with y% of the principal. Suppose that people are risk aversion, and the utility function is a logarithmic utility function, that is, U=lg (w), w is the finance after gambling, and the utility in different states can be directly added. Then the mathematical expectation after a game is:

$$\frac{1}{2}\lg 100(1-x-y+3x)+\frac{1}{2}\lg 100(1-x-y+4y)$$
$$=\frac{1}{2}\lg 10^4(1+2x-y)(1-x+3y)$$

In order to maximize the benefits, we only need to maximize the mathematical expectation that is, the above expression, and then calculate the maximum value of the derivative of the function.

Solution:

$$f(x,y)=(1+2x-y)(1-x+3y)$$
$$f'_x(x,\ y)=2(1-x+3y)-(1+2x-y)=1-4x+7y=0$$
$$f'_y(x,\ y)=-(1-x+3y)+3(1+2x-y)=2+7x-6y=0$$
$$\Rightarrow\begin{cases}4x-7y=1\\7x-6y=-2\end{cases}\Rightarrow\begin{cases}x=-\dfrac{4}{5}\\y=-\dfrac{3}{5}\end{cases}$$

therefore, there is a maximum in the closed region $D=\{(x,y)\mid 0\le x+y\le 1,0\le x\le 1,0\le y\le 1\}$

Compare the function value of the maximum and the boundary point, and the maximum point is $x=\frac{1}{2}$, $y=\frac{1}{2}$, So you should bet 50 dollars to guess the odd number and 50 yuan to guess the even number. It is an interesting answer!

The above scenario can be generalized, that is, you can bet k dollars at will. If you guess odd numbers 1, 3, 5, you will win a times of k dollars, and if you guess even numbers 2, 4, 6, you will win b times of k dollars. It is also stipulated that you can bet odd numbers and even numbers at the same time. You can use the same method like before to get the following conclusions:

$$\begin{cases} b \le 2, & \text{nothing} \\ \frac{1}{a}+\frac{1}{b} \le 1, & \text{invest}B \\ \frac{1}{a}+\frac{1}{b} \ge 1, & \text{invest}50\%A, 50\%B \end{cases}$$

This indicates that in general, we should bet all funds to maximize long-term benefits. Let's take a look at an interesting little question again. A tetrahedron screen is printed with four numbers: 0, 1, 2, and 3. Players can bet at them will and determine the winning multiple by the number of points. (For example, if you bet 10 dollars and the number of points thrown by the sieve is 2, you will get 10 × 2=20 dollars), assuming there is 100 dollars, how much do you intend to bet?

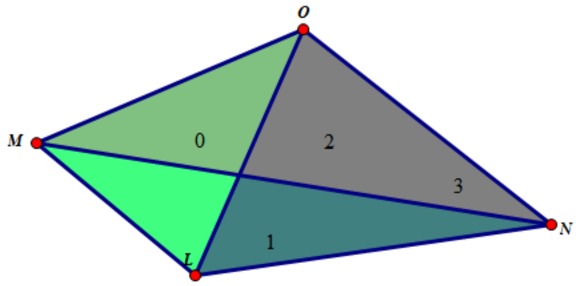


**Fig 1.** Fun tetrahedral screen game

if you bet 100 dollars, there are ¼ chance may get 300 dollars, but also you may lose all your money at the same time. So invest all your money is not a good way. If you don't want to take risks and bet too little, then the growth rate of funds is very low. Similarly, we solve the problem under the condition that the utility function adopts a logarithmic function, We can use the method of calculating the mathematical expectation of logarithmic principal after a round and calculating the maximum value by derivative is still adopted like before. Let Assumed that x% of the principal is invested, and then the mathematical expectation of logarithmic principal after a round is:

$$\frac{1}{4}\lg 100(1-x)+\frac{1}{4}\lg 100(1+0\times x)+\frac{1}{4}\lg 100(1+x)+\frac{1}{4}\lg 100(1+2x)$$
$$=\frac{1}{4}\lg 10^3(1-x)(1+x)(1+2x)$$

In order to maximize the income, we only need to maximize the mathematical expectation, that is, the above expression.

Solution:Let

$$f(x)=(1-x)(1+x)(1+2x)=1+2x-x^2-2x^3$$

$$f'(x) = 2 - 2x - 6x^2 = 0$$
$$\Rightarrow x_1 = \frac{-1+\sqrt{13}}{6}, x_2 = \frac{-1-\sqrt{13}}{6}$$

That is, about 43.4 dollars should be invested.

These three examples show a calculation method to maximize the long-term growth rate and benefit, that is, to maximize the mathematical expectation through calculation to achieve the goal, which coincides with the idea of Kelly formula. In other words, Kelly formula is derived based on this idea.

Summarize the above methods for solving the problem,We can summarize the expression of the above example in the following form: assuming that the principal is c, the investment ratio is x, and b is the net loss ratio, the probability of winning the game is p, and the probability of losing the game is q, it is equal to1-p, then the income after N games is: $f(x) = c\ (1+bx)^{Np}(1-x)^{N(1-p)}$,In order to facilitate the calculation of maximum value by derivative, the logarithmic derivative method is well adopted.

$$\ln f(x) = \ln c + Np\ln(1+bx) + N(1-p)\ln(1-x)$$

Derive from the right side of the equation:

$$\frac{bNp}{1+bx} - \frac{N(1-p)}{1-x} = 0$$
$$\Rightarrow x = \frac{bp-q}{b}$$

This is the expression of Kelly's formula. Next, we will introduce the numerical values of the first game into the Kelly formula to verify its correctness. Substitute the value in Scenario 1 into the calculation, that is, b=2, p=q= ½, It is calculated that x=25%, which is consistent with our previous results.

The above examples only consider the possibility of a very small number of different outcomes in the game outcome. In fact, the game or investment problem is faced with many situations. We need to generalize the above formula to get a formula with a wider application range.

We can generalize the above three examples, expand the two results in the game into n results and express them in the following form: let the gambler's principal be M, the investment ratio be x, the game has n results (n >= 2) per game, the net yield of the ith result is, the probability of the ith result is, then the mathematical expectation of logarithmic principal increment after one game is

$$f(x) = \sum_{i=1}^{n} p_i \lg[M(1+xr_i)] - \lg M = \sum_{i=1}^{n} p_i \lg(1+xr_i)$$
$$f'(x) = \sum_{i=1}^{n} \frac{p_i r_i}{1+fr_i} = 0$$

It shows that the long-term benefit can be maximized if the investment ratio meets the above expression. It can be seen that the example of "roll dice" is the special form of this expression. In fact, let the

$$n = 2,\ p_1 = \frac{1}{2},\ p_2 = \frac{1}{2},\ r_1 = -1,\ r_2 = 2$$

And bring it in the above expression,you can get the answer 25%,it is also the answer to Scenario 1.

Specially, let

$$n = 2,\ q = 1-p,\ r_1 = -1,\ r_2 = b$$

B is the net loss ratio, then the mathematical expectation of the logarithmic principal increment obtained by substituting the above expression is:

$$f(x) = p\lg(1 + xb) + q\lg(1 - x)$$

$$f'(x) = \frac{bp}{1+bx} - \frac{q}{1-x} = 0 \Rightarrow x = \frac{bp-q}{b}$$

This is the expression of Kelly's formula. We can bring the values in Scenario 1 into the Kelly formula for checking, and find that the answer is the same as the values told by Kelly formula.From this we derive a more general Kelly expression.

Under the background of scenario 1, using Kelly formula, b=2, p=q=50%, 25% is calculated. The following is the mathematical simulation, which uses the method of randomly generating a series of numbers to simulate the dice experiment in excel. The experiment has been carried out for 20 times, and in fact, it can be done all the time. The experimental data shows that with the increase of the number of experiments, the income obtained by betting 25% is better and better than others. Relevant data of the experiment are as follows:(Experimental simulation of investment with different investment ratios in a one out of three scenario)

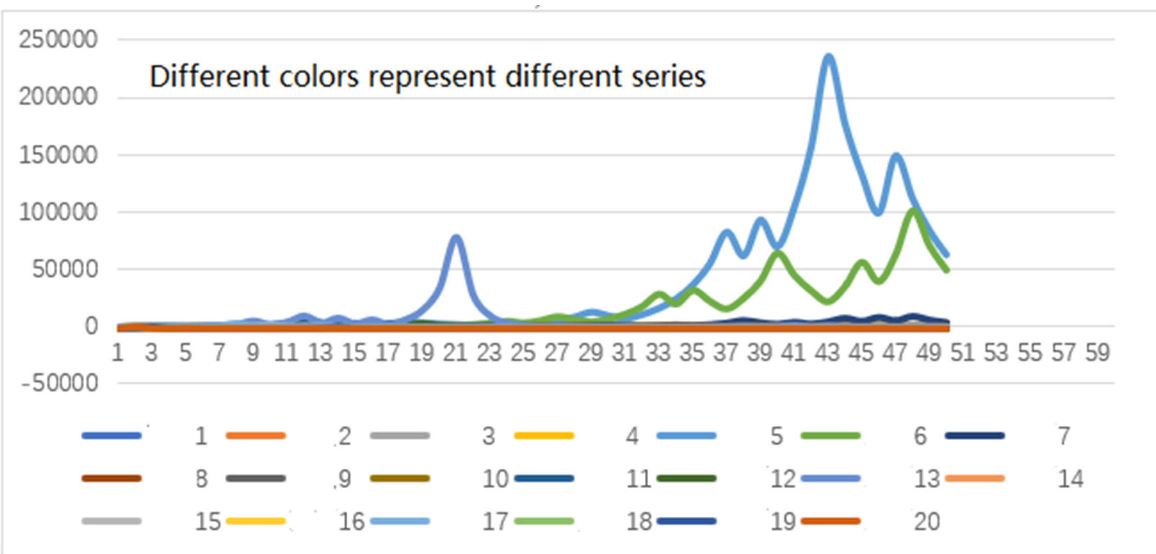


**Fig 2.** Relevant data of the experiment

(Category 1 represents an investment ratio of 0.1%, followed by an increase of 0.05%.)

According to the experimental data, it is not difficult to find that although the option of investing 25% in a short period of time seems to have little advantage, or even lag behind, in the long run, this option is the best choice, because with the increase of the number of trials, we can find that the probability of success and failure is closer to 50%, in fact, it is subject to normal distribution, so we predict that with the increase of the number of trials, Compared with other investments, the 25% advantage of investment is increasing.

## 3. The Application of Kelly Formula

After verifying the Kelly formula, let's take a specific look at its application in the real estate and stock markets.When the housing market is rising or falling, we can reasonably use the Kelly formula to decide the funds to be invested or withdrawn. This paper takes Beijing's housing price as an example to illustrate the application of Kelly's formula in the housing market.In the period of rapid rise of housing prices in Beijing, through a simple analysis of the market, we assume that there is a

70% probability of a 50% increase, and a 30% probability of returning to the original, using Kelly's formula, p=0.7, q=0.3, b=0.5, and substituting the formula to calculate x=10%, which shows that investing 10% of the capital into the housing market can maximize the income. In fact, through a specific analysis of housing prices in Beijing in recent years, we can calculate that leverage investment should be added through the Kelly formula. Next, we will look at the application of Kelly's formula in the stock market.In fact, the rise and fall of stocks are similar to the rise and fall of the housing market. We can also use the above method to calculate the funds invested or withdrawn from the stock market. This paper takes the Chinese stock market as an example to illustrate the application of Kelly formula in the stock market.According to the analysis of the stock market, there are 60% opportunities to rise from 1000 points to 2000 points, and 40% opportunities to fall from 1000 points to 500 points. According to the Kelly formula, 90% of the investment is required.From the analysis of the above two examples of the housing market and the stock market, we can know that Kelly formula is very suitable for the management of funds and the allocation of resources. In fact, Kelly formula is also suitable for portfolio investment. It can be seen that reasonable use of Kelly formula can maximize our long-term capital income.

## 4. Conclusion

Starting with three typical examples of rolling dice, we found a method to maximize long-term benefits. This method is the main idea of Kelly formula and that means we get the maximum mathematical expectation of logarithmic returns. And then we use this optimization idea and method to derive and generalize Kelly formula, and verify the rationality of Kelly formula with experiments. By using Kelly formula, we not only can calculate the amount of money invested or withdrawn in the housing market or the stock market, but also can manage our money. In general, Kelly formula can be widely used in fund management, portfolio investment and other applications. In fact, the Kelly formula also has certain limitations, such as when it fails when there are negative expectations, and in real life, the mathematical expectations of each investment change, which greatly reduces the accuracy of the Kelly formula.

## References

[1] On January 21, 2019, the investment secret of Buffett's long bet must win - Kelly formula, personal library website, the investment secret of Buffett's long bet must win - Kelly formula (360doc. com).

[2] Kelly formula quoted by Buffett, Sohu. com, June 21, 2016, "Kelly formula" quoted by Buffett: It is not easy to say love you (sohu. com).

[3] One of the most important formulas in investment -- Kelly formula, Zhihu, and one of the most important formulas in investment -- Kelly formula, Zhihu (zhihu. com).

[4] The application model of Kelly formula in stock portfolio investment.