Study of Optimal Portfolio Performance on Markowitz and Index Model

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Abstract. The choice of a portfolio optimization of shares has been for so long a subject of major interest in the field of investment and financial mathematics. This article aims to create a portfolio optimization which follows the objective of maximized return and minimized risk under given constraint among S&P 500 and the ten stocks, utilizing data collected from Yahoo! Data. The article ought to elucidate the Markowitz Model and Single Index Model in terms of generating a better result via comparing their distinct mechanisms in calculating their return percentage, standard deviation and Sharpe ratio, alongside with the corresponding efficiency frontier to compare their sharp ratio to determine a better model for each scenario. The result illustrates a better performance of the Markowitz Model in the aspect of producing a better portfolio optimization contrasting the Index Model in all the constraint except constraint three. Eventually backtesting was utilized to verify the feasibility.

Keywords: Markowitz model; index model; portfolio optimization; Sharpe ratio.

1. Introduction

A The choice of a portfolio optimization of shares has been a subject of major interest in the field of investment and mathematics. It is an analytical process of selecting and distributing a group of investment assets in which the portion of allocated investment is distributed into a portfolio that have an optimized expected return and minimal risk. The concept of a portfolio optimization first came out in 1952, when Markowitz raised up a popular model called the mean-variance model that is used to calculate stock portfolio weight. Given the expected value at risk, the M-V model is utilized to find the best portfolio that maximizes return on an investment and reduce the upcoming risks. For instance, investors is aiming for two objectives for a given portfolio with weight vector, they are: (i) Maximizing the expected value μ of a portfolio return, and (ii) Minimizing the portfolio risk measured by σ [1]. Sharp then simplified Markowitz's model and proposed the single index model, which aimed to link an excess market portfolio with the risk and return of the portfolio, which provides a new perspective for portfolio optimization, by greatly cuts the amount of calculation to run the model [2].

The following studies did some comparison between either the Markowitz Model or Index Model and with other models. Putra utilized Indonesia Stock Exchange to make a comparison between Single Index Model and Markowitz Model on their portfolio optimization performances, the results illustrate that conservative type of investors have a natural tendency to avoid risk can invest in portfolios formed by Index Model. Dichotomy, investors that are aggressive can invest in portfolios formed by the Markowitz Model [3]. Yuwono and Ramdhani utilized Jakarta Islamic Index to make a comparison in optimal portfolio formation between the Markowitz and Index Model through the method of quantitative approach, and they have concluded that there is no significant difference between the results being produced, and they have suggested to use the concept of downside risk in post modern portfolio theory to form future researches [4]. Chasanah and Lesmana used M-V Criterion in optimal portfolio formation, also using Jakarta Islamic Index to compare the Markowitz and Index Model. Eventually they concluded that the portfolio optimization formed with Markowitz Model performed better than the one formed with Single Index Model based on Jakarta Islamic Index [5]. Bartkus and Paleviciene compared the Markowitz Model and the Minimax Model, and compared the two Model's linear dependence between macroeconomic indicator changes and share's return.

Ultimately they concluded that the Markowitz and MAD Model is more suitable for conservative investors due to they are less dependent on social and macroeconomic changes [6].

Furthermore, study from Chaweewanchon and Chaysiri tried to improve the performance of Markowitz Model by applying a prediction method using hybrid machine learning model for portfolio formation to ensure a better stock input, which shows that preselection of stocks can improve MV performance, which is proven by the results that they outperform comparison models in terms of Sharpe ratio, mean return, and risk [7].

In addition to this, there are also other studies that did comparison and analysis on different strategies and approaches using different methodologies and models to form portfolio optimization. For instance, Amiri and Alimi conducted a study using the Invasive Weed Optimization (IWO) algorithm that calculates the return and risk of each asset based on cardinality constraint and investment income per share to form a model to generate portfolio optimization, which also found out to be able to successfully generate an efficient optimal portfolio formation [8]. Guennoun, and Hamza presented an approach to reach portfolio optimization based on the classification, genetic algorithms and the algorithm MinVaRMaxVaL to obtain a portfolio optimization of a decreased size from the previous portfolio, which results in a financial gain surplus in terms of cost and taxes reduction, and a performance at design loads reduction.[9] Mazraeh et al. set up stock portfolio optimization using a combined approach method. Technical analysis analyzed the statistical trends collected from trading activities and sets up six indicators and six levels to filter the stock companies. Markow Chain examined groups of data based on the time chronology of the data and the calculation of the amount of increment or decrement of the average price. They also utilized the Machine Learning and Random Forest method to calculate the correlation between stocks, which ends up in SVM learning machine method leading to a small correlation error than the random forest method. The author eventually formed an optimal portfolio based on two algorithms, MOGWO and NSGA II. And the results shown that shows that the MOGWO algorithm has higher efficiency in stock portfolio optimization [10].

Subsequently, after reading all these papers that holds diverse opinions towards the two models and portfolio optimization, it is best to go through Markowitz Model and Index Model's exact process of performing a portfolio optimization to substantiate their true effectiveness and make a comparison between the two methods to determine which one of them hold the best portfolio performance.

2. Methodology

2.1. Model Theory Comparison

2.1.1 Markowitz model

The Markowitz mean-variance model is based on the following assumptions: Investors are always rational and will try to look for a maximized return while minimizing the risk. Investor's choices are solely dependent on return and risk, disregarding outside cost factors such as tax or transaction cost. All assets are infinitely divisible. Investors estimate portfolios based on the mean and standard deviation of the expected return.

The expected return of MM model portfolio P is:

$$R_p = \sum_{i=1}^{n} w_i r_i \tag{1}$$

The Standard deviation of Portfolio is:

$$\sigma_p = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i \, w_j cov(R_i, R_j)} \tag{2}$$

Here, ri represents Asset I's expected return, wi represents the proportion of asset i in the portfolio, n refers to number of all assets, and $COV(Ri\,, Rj)$ represents the covariance between the return of asset i and asset j.

2.1.2 Index model

The Index Model is based on the following assumptions: One single factor primarily influences individual security returns. The security's return can be explained by its relationship with the market return. Any return that is not explained by the market return is due to the unique or firm-specific risk, which is assumed to be uncorrelated across different securities. These idiosyncratic (or non-market) risks are unsystematic and can be diversified away while systematic (market) risks are unavoidable. The previous assumptions indicates that $\mathrm{cov}(\mathrm{R_m}, \varepsilon_\mathrm{i}) = 0$; $\mathrm{cov}(\varepsilon_\mathrm{i}, \varepsilon_\mathrm{j}) = 0$. The expected return of the IM model portfolio p is:

$$R_p = \sum_{i=1}^{n} w_i r_i \tag{3}$$

The Standard deviation of Portfolio P is:

$$\sigma_p = \sqrt{(\sum_{i=1}^{n} w_i \beta_i)^2 \sigma_M^2 + \sum_{i=1}^{n} w_i^2 \sigma_\varepsilon^2} \tag{4}$$

Excess rate of Return is:

$$R_i(t) = \alpha_i + \beta_i R_M(t) + e_i(t) \tag{5}$$

Here, r_i stands for Asset I's expected rate of return, wi stands for the proportion of asset i in the portfolio, n refers to the number of all assets, β_i is the factor of risk for asset I, σ_M refers to systematic risk, σ_ε is the unsystematic risk.

2.2. Selection of Data

Analysis of daily total return from historical data for 10 stocks over the last 20 years was conducted, along with the S&P 500 and the one month federal funds rate. The ticker symbols for the 10 stocks are AAPL, CTXS, JPM, JBHT, PGR, AMZM, UPS, FDX, LSTR, BRK/A. These data are from Yahoo! Data and the above stocks belong to the consumer cyclical, technology, financial services and industrials sectors.

To reduce the non-Gaussian effects of the data values, daily data was converted into the monthly data by simply systematically used the last day's value of each month as monthly data. After this coarse-graining of the historical data, the confirmation of the improvement in Gaussian properties in the historical data was achieved. Figure 1, 2, 3 below provides a visualization of how FDX, AMZN, and UPS's share price was testified obeyance of the normal distribution, and thus the monthly data was selected to be used in the following calculations.

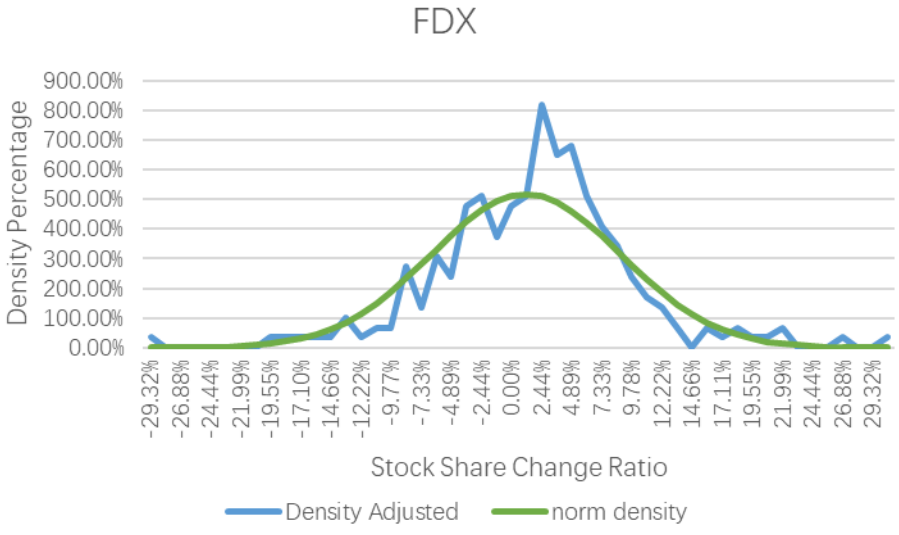


Fig. 1 Normal Distribution graph for FDX's Share Prices

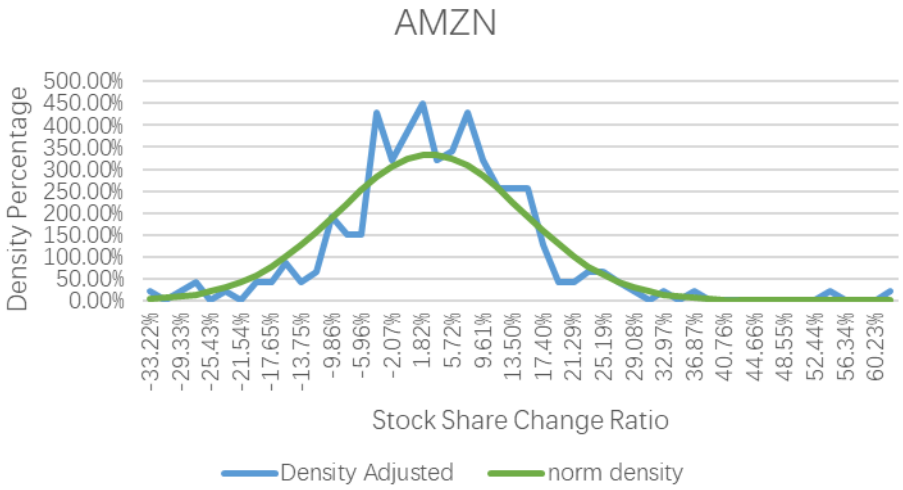


Fig. 2 Normal Distribution graph for AMZN's Share Prices

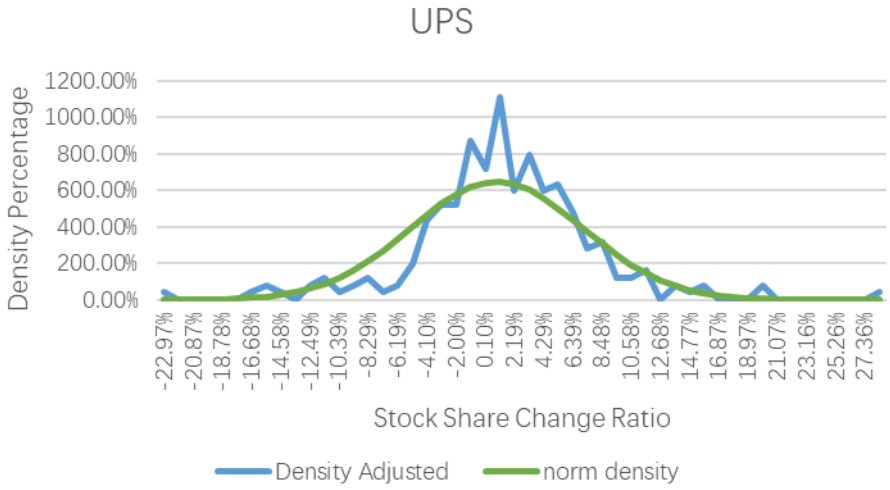


Fig. 3 Normal Distribution graph for UPS's Share Prices

2.3. Conceptual Framework

Before proceeding with the data processing, some basic concepts is introduced: excess return, average excess return, standard deviation, residuals, and Sharpe ratio. Excess return is the risk premium for a given holding period. The risk premium is the contrasting value between the expected return over the holding period and the risk-free rate. This article uses the one-month federal funds rate as a proxy for the risk-free rate. The average excess return indicates the average level of security returns above the risk-free rate. Standard deviation was used as a complete measure of risk. The residual is the difference between the actual return and the model's prediction and tests the reliability of the model to some extent. In general, there is a positive correlation between the expected return and risk. The Sharpe ratio is the ratio of return to risk. In addition, some factors to analyze the model was also needed, for the Markowitz model we need average return, standard deviation and correlation, for the index model we need to calculate its bata, alpha and residual standard deviation, etc.

2.4. Additional Constraints

In order to make the model more realistic, four additional constraints are used in this paper in the calculation of the two types of portfolios, which together with the unconstrained case make a total of five different additional constraint cases. It is required to optimize the portfolios under the following constraints:

The Constraint 1: this constraint is aimed to simulate the Regulation T by FINRA, by allowing broker-dealers to allow their customers to have positions, 50% or more of which are funded by the customer's account equity.

$$min\,\sigma_P\,,\ st\ \sum_{i=1}^{11}|w_i| \leq 2 \tag{6}$$

The Constraint 2: This restriction is formed to simulate some random "box" constraints on weights, which can be provided by the client:

$$min\,\sigma_P\,,\ st\ |w_i| \leq 1,\ for\ \forall i \tag{7}$$

The Constraint 3: A "free" problem, without any additional constraints, to testify the situation without any market constraints:

$$min\,\sigma_P\,, no\ constrains \tag{8}$$

The Constraint 4: This restriction is designed to simulate a typical limitation of an open-ended mutual fund that forbids any short positions existing in the U.S.:

$$min\,\sigma_P\,, stw_i \geq 0, for\ \forall i \tag{9}$$

The Constraint 5: This restriction is introduced to investigate if the inclusion of the broad index into our portfolio has whether positive or negative influence.

$$min\,\sigma_P\,, stw_1 = 0 \tag{10}$$

3. Results and Discussion

3.1. Data Calculation

There are several procedures need to be conduct to the raw data. The data first is aligned with the vlookup function to modify into monthly returns that best fits the normal distribution. Then excess return and residual excess return is computed, also the monthly return which is calculated by end date price divided by start date price. Next excess return is calculated by the return of stock minus return of risk minus free funds. Then residual of excess return equals to excess return of stock minus excess return of SPX * beta - alpha/12. Next involves the calculation of annual average return, annual standard deviation, beta, alpha and residual standard deviation by using STDEV.P, AVERAGE, SLOPE, INTERCEPT function in excel. Then, apply the correlation function of data analysis inside the data column in excel, a matrix plot of correlation can be obtained, which is illustrated by Figure 4. From this matrix diagram, it can be seen that correlations among stocks within the same industry are high and among stocks from different industry are low. SP500 has a high correlation with different stocks.

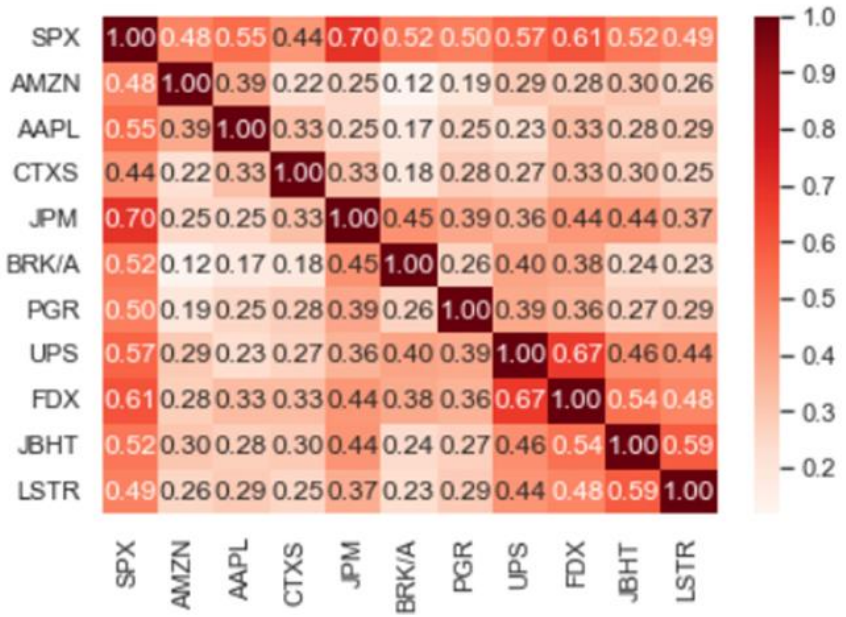


Fig. 4 Correlation Analysis

Eventually, the solver function in data is applied to analyze the Markowitz model and the Index Model by setting the target and changing the variables, and setting the constraints for different scenarios to obtain the data illustrated in Table 1 and Table 2, so that the weighting of each stock in the portfolios under the constraints of the different models and the expected return of each portfolio is presented.

Table 1. Solver table for Markowitz Model in Percentage

MM (Cons1)	SPX	AMZN	AAPL	CTXS	JPM	BRK/A	PGR	UPS	FDX	JBHT	LSTR	Return	Sharpe
MinVar	72	-2.3	-3.8	-1.0	-18	36.2	13	3.4	-10	-0.5	10.8	7.1	58.4
MaxSharpe	-48	16.4	30.0	-0.1	-0.1	41.3	33	0.0	-1.5	12.5	16.7	26.4	141
MM (Cons2)	SPX	AMZN	AAPL	CTXS	JPM	BRK/A	PGR	UPS	FDX	JBHT	LSTR	Return	Sharpe
MinVar	72	-2.3	-3.8	-1.0	-18	36.2	13	3.4	-10	-0.5	10.8	7.1	58.4
MaxSharpe	-100	22.3	39.8	-1.2	-0.5	62.5	46	-3.1	-10	20.9	23.9	33.2	150
MM (Cons3)	SPX	AMZN	AAPL	CTXS	JPM	BRK/A	PGR	UPS	FDX	JBHT	LSTR	Return	Sharpe
MinVar	72	-2.3	-3.8	-1.0	-18	36.2	13	3.4	-10	-0.5	10.8	7.1	58.4
MaxSharpe	-237	37.1	65.4	0.4	17	91.6	68	1.3	-8.7	31.0	34.0	49.6	153.9
MM (Cons4)	SPX	AMZN	AAPL	CTXS	JPM	BRK/A	PGR	UPS	FDX	JBHT	LSTR	Return	Sharpe
MinVar	38	0.0	0.0	0.0	0.0	38.5	14	0.7	0.0	0.0	8.1	10.0	76.7
MaxSharpe	0.0	13.0	25.2	0.0	0.0	19.3	22	0.0	0.0	8.8	11.0	22.1	125.4
MM (Cons5)	SPX	AMZN	AAPL	CTXS	JPM	BRK/A	PGR	UPS	FDX	JBHT	LSTR	Return	Sharpe
MinVar	0.0	2.5	4.2	0.8	-7	56.3	23	11	-8.1	0.6	15.8	13.2	98.6
MaxSharpe	0.0	14.7	26.7	-3.4	-15	36.3	32.0	-12	-13	17.8	16.8	23.9	132.6

Table 2. Solver table for Index Model in Percentage

IM (Constr1):	SPX	AMZN	AAPL	CTXS	JPM	BRK/A	PGR	UPS	FDX	JBHT	LSTR	Return	Sharpe
MinVar	65.8	-4.1	-4.7	-2.4	-12.9	34.5	13.4	8.6	-3.6	-1.7	7.2	6.5	52.0
MaxSharpe	-47.3	17.8	30.5	0.5	-2.7	22.4	31.5	1.2	0.1	19.2	26.7	28.6	143.0
IM (Constr2):	SPX	AMZN	AAPL	CTXS	JPM	BRK/A	PGR	UPS	FDX	JBHT	LSTR	Return	Sharpe
MinVar	65.8	-4.1	-4.7	-2.4	-12.9	34.5	13.4	8.6	-3.6	-1.7	7.2	6.5	52.0
MaxSharpe	-100.0	21.8	36.6	3.1	-8.8	33.3	40.0	9.5	5.6	25.0	33.8	34.1	152.1
IM (Constr3):	SPX	AMZN	AAPL	CTXS	JPM	BRK/A	PGR	UPS	FDX	JBHT	LSTR	Return	Sharpe
MinVar	65.8	-4.1	-4.7	-2.4	-12.9	34.5	13.4	8.6	-3.6	-1.7	7.2	6.5	52.0
MaxSharpe	-329.8	43.0	70.0	11.0	8.8	58.5	72.3	27.9	24.9	50.3	62.9	60.9	159.6
IM (Constr4):	SPX	AMZN	AAPL	CTXS	JPM	BRK/A	PGR	UPS	FDX	JBHT	LSTR	Return	Sharpe
MinVar	30.5	0.0	0.0	0.0	0.0	37.7	14.6	9.3	0.0	0.0	7.9	10.2	78.8
MaxSharpe	0.0	15.5	27.1	0.0	0.0	3.3	21.4	0.0	0.0	14.0	18.8	24.5	127.7
IM (Constr5):	SPX	AMZN	AAPL	CTXS	JPM	BRK/A	PGR	UPS	FDX	JBHT	LSTR	Return	Sharpe
MinVar	0.0	-1.6	-0.8	-0.1	-5.0	48.0	21.6	17.8	3.5	2.8	13.8	11.5	86.9
MaxSharpe	0.0	18.6	31.7	-0.5	-21.2	11.7	27.7	-5.3	-5.4	18.6	24.2	26.9	133.1

3.2. Efficient Frontier

In order to better demonstrate the difference in portfolio optimization generated by the two model, the efficiency frontier is utilized to visualize the numerous data value, in which it represents all possible portfolios that provides an expected return for a given level of risk.

Constraint 1: The Markowitz and index model in restriction of condition 1 is illustrated in Figure 5 and Figure 6, depicting the capital market line and minimum variance frontier. The CAL of the

Markowitz model and the risky asset portfolio are above the Index Model, with a rate difference of 1.04% between their expected returns, which decrease as risk increases. This is due to the weakening of residual item correlation between different stocks when weight is greater than or equal to 50%, resulting in minimal differences between the two model.

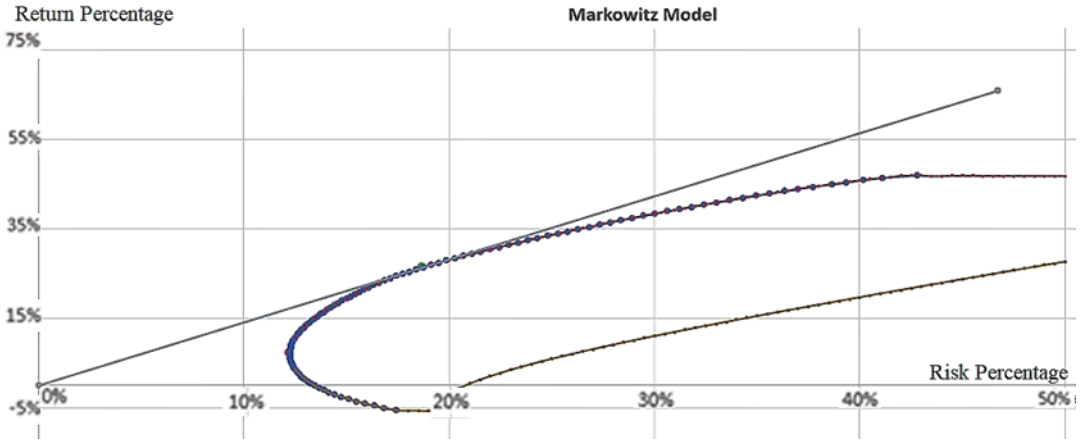


Fig. 5 Markowitz Model Efficient Frontier, constraint 1

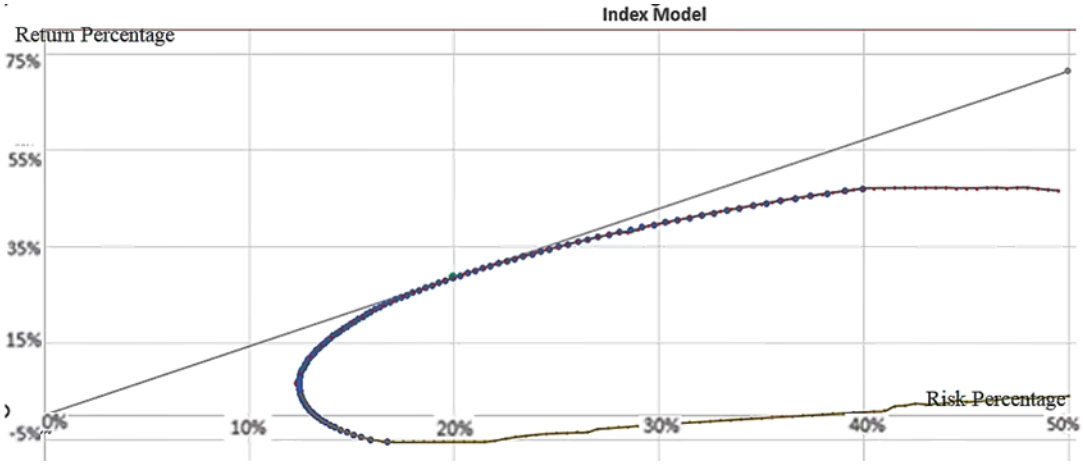


Fig. 6 Index Model Efficient Frontier, constraint 1

Constraint 2: Figure 7 and Figure 8 below displays the minimum variance boundary and capital market line for both the Markowitz model and the index model, subject to condition 2. Notably, the Markowitz model's CML and risky asset portfolio exceed those of the index model, with a 3.79% difference in expected returns that increases with risk. This discrepancy arises due to the arbitrary "box" constraints on weights, which do not significantly differ from unconstrained weights and result in residual term correlations across stocks. The index model overlooks this correlation, while the Markowitz algorithm accounts for it when minimizing portfolio variance, leading to a superior optimal portfolio compared to the single index model.

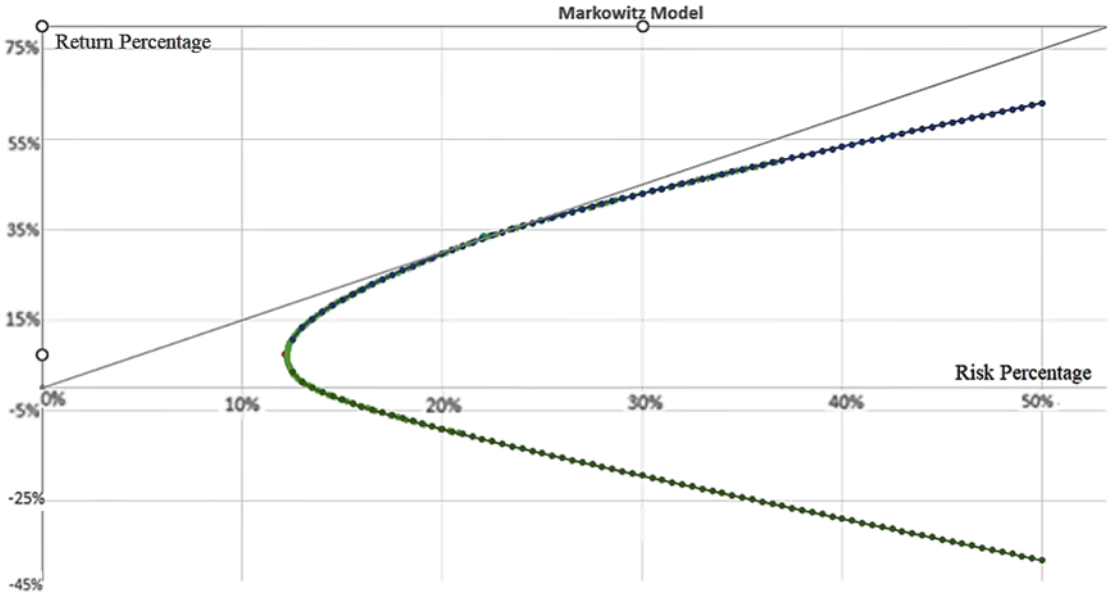


Fig. 7 Markowitz Model Efficient Frontier, constraint 2

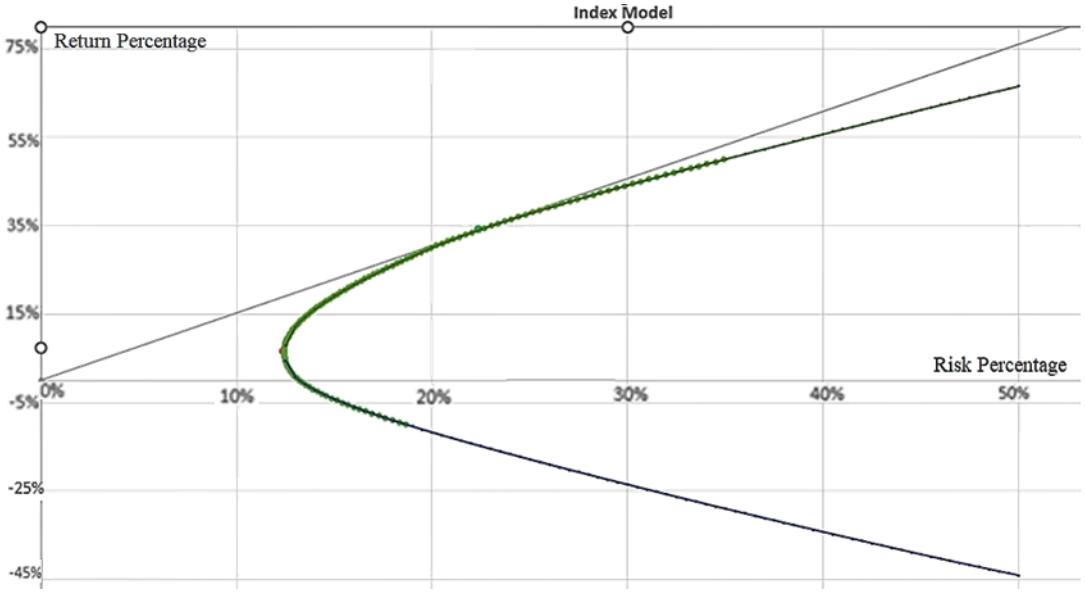


Fig. 8 Index Model Efficient Frontier, constraint 2

Constraint 3: The two model with the restriction of condition 3 are illustrated in the Figure 9 and Figure 10, displaying the capital market line and the minimum variance frontier. The CML of the Markowitz model and the risky asset portfolio both surpass the index model, with identical expectations. As risk rises, the yield spread widens to 4.02%. This can be attributed to the correlation of residual terms between various stocks, which is disregarded by the Index Model under normal conditions. The Markowitz algorithm, on the other hand, automatically minimizes portfolio variance while considering this correlation, leading to a superior optimal combination compared to the single Index model.

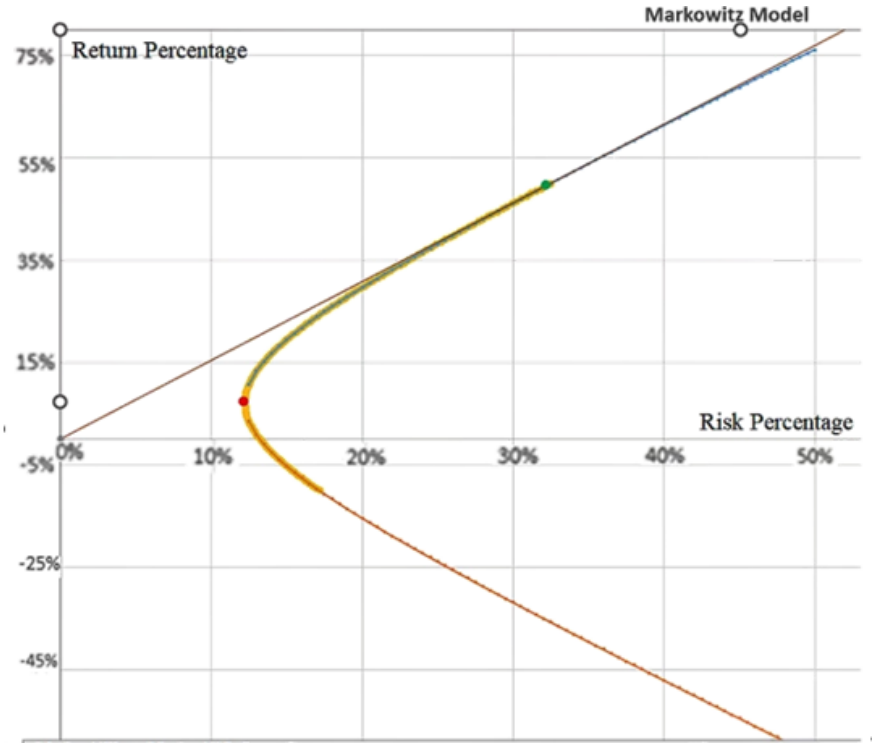


Fig. 9 Markowitz Model Efficient Frontier, constraint 3

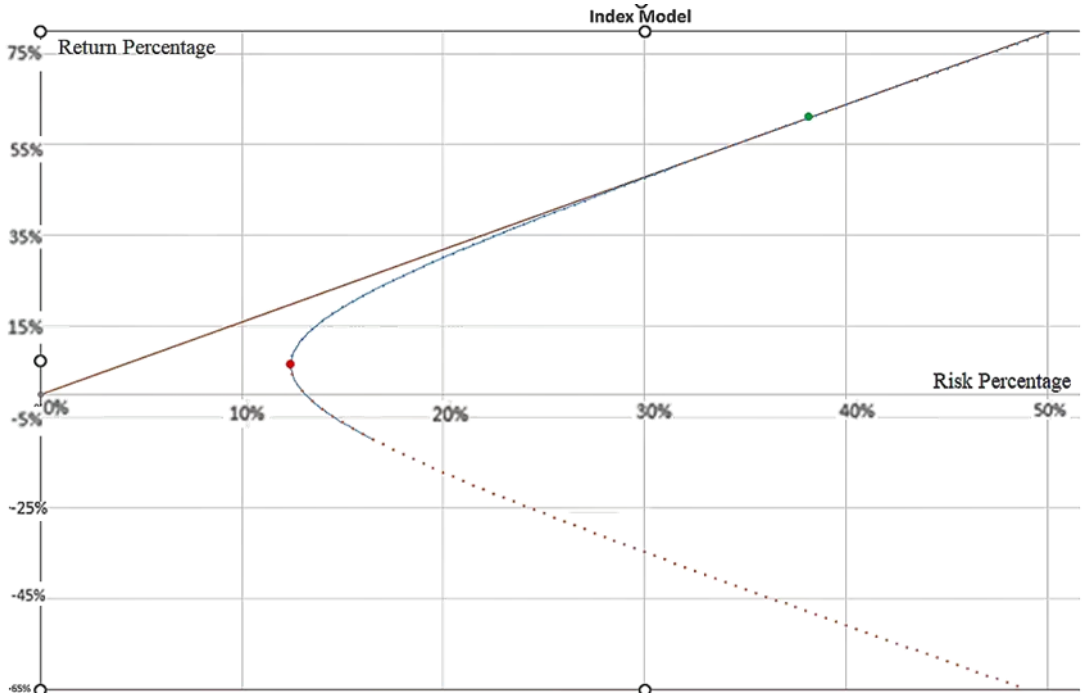


Fig. 10 Index Model Efficient Frontier, constraint 3

Constraint 4: The Markowitz model and Index model with constraint 4 can be demonstrated by Figure 11 and Figure 12. The CML of the Markowitz model and the risky asset portfolio showed not much difference compared to the Index Model. This is because when short selling is not allowed, the correlation between the residual terms of different stocks is weakened, which also weakens the difference between the results generated by the two models.

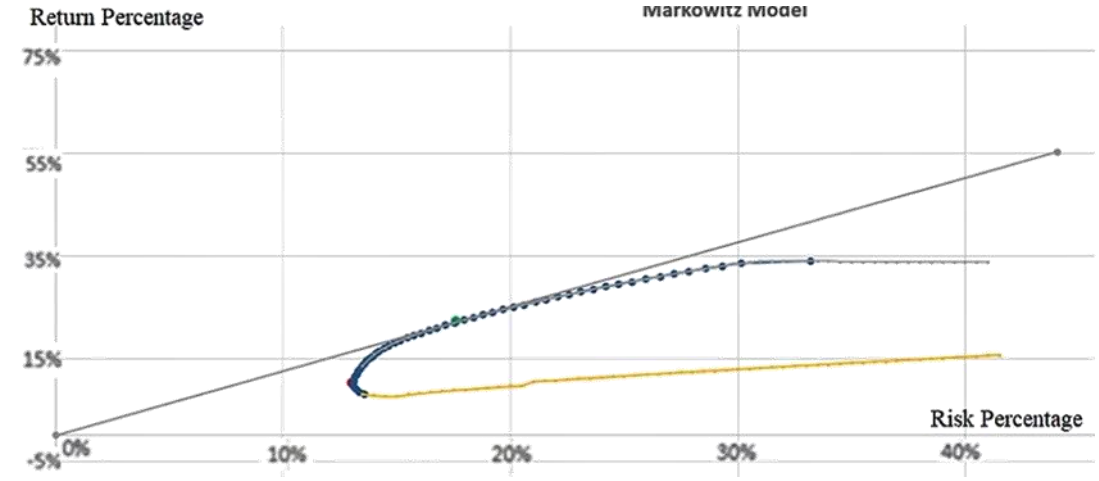


Fig. 11 Markowitz Model Efficient Frontier, constraint 4

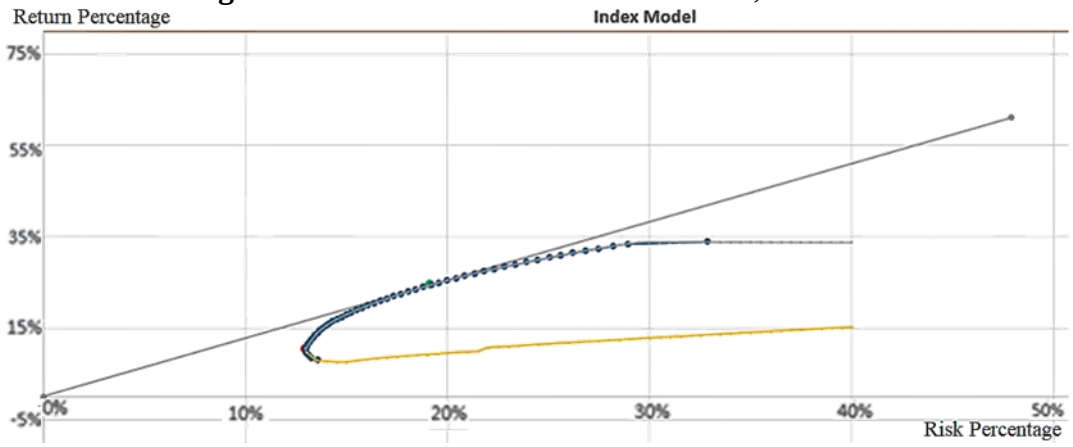


Fig. 12 Index Model Efficient Frontier, constraint 4

Constraint 5: The two model with the restriction of condition 3 are illustrated in the Figure 13 and Figure 14, displaying the minimum variance frontier and the capital market line. The CML of the Markowitz model and the risky asset portfolio both surpass the index model, the difference of expected return rate between the two models is 2.3%, which increases with the increase of risk. This is because the stocks related to the residual term account for a large proportion of the entire portfolio, eventually makes Markowitz Model to have a better result than Index Model.

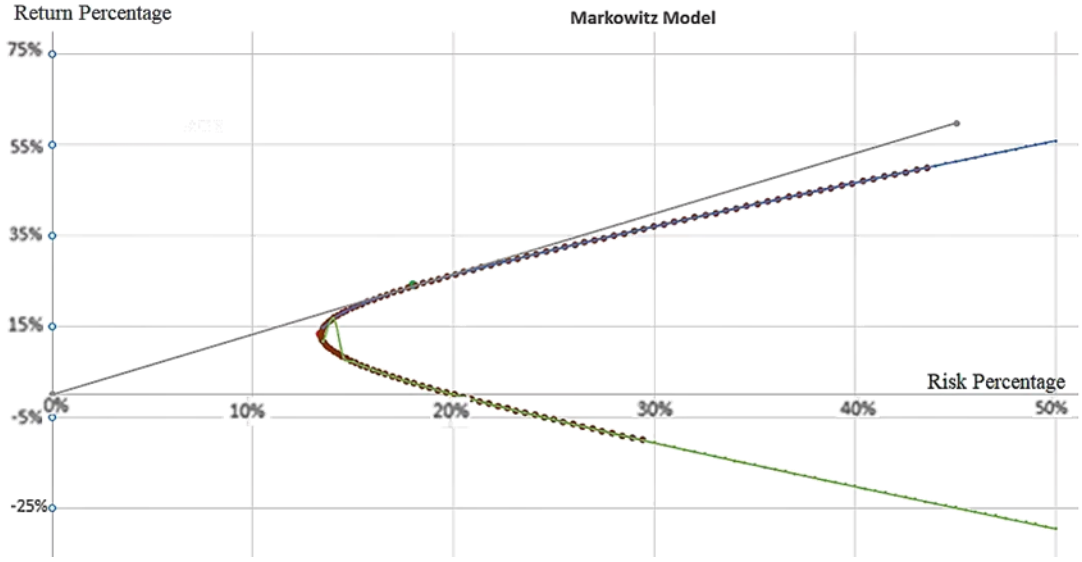


Fig. 13 Markowitz Model Efficient Frontier, constraint 5

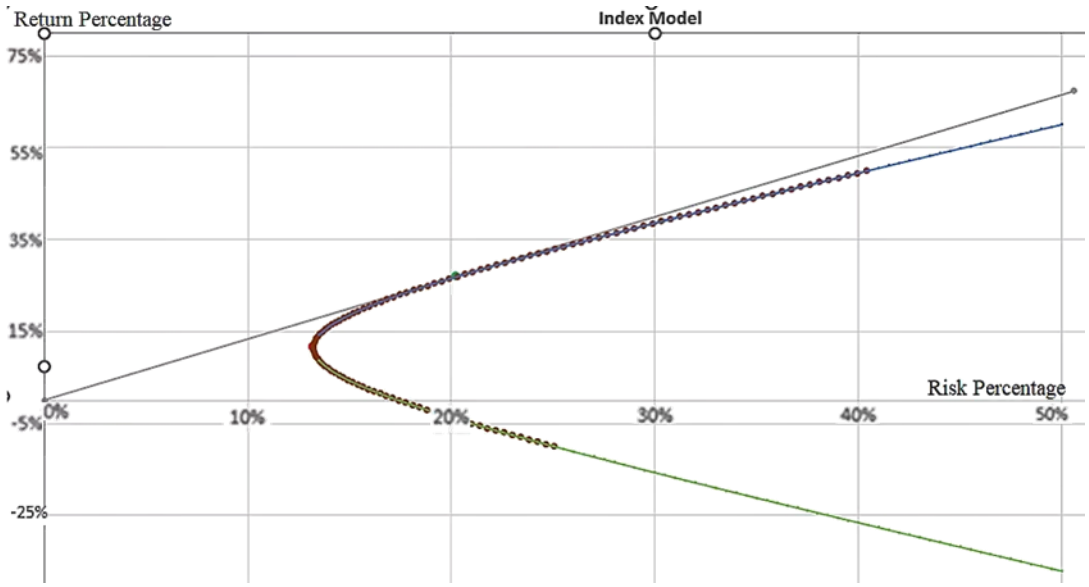


Fig. 14 Index Model Efficient Frontier, constraint 5

3.3. Backtesting

It is best to use the future data to test the model to verify the feasibility, but only previous data was available, so the data of 2000 year was used to replace the future data and bring it into the existing portfolio.

Backtesting can be used to test whether the above model is feasible. The way to do this is to multiply the monthly return of the previous year's stock data by the min var parameter of the existing portfolio under the Markowitz Model theory and do a line plot to see if its value is greater than the SPX return of the next year.

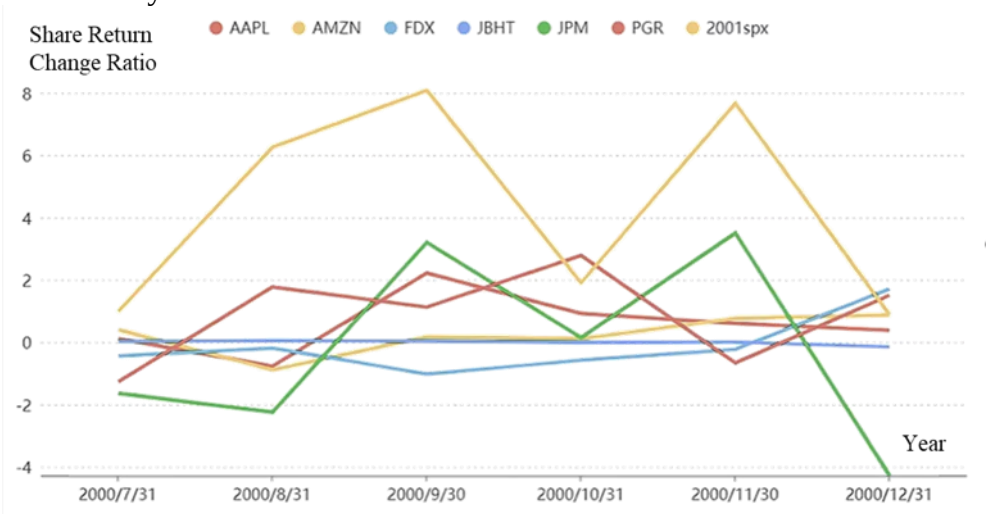


Fig. 15 Stock Returns by Percentage, backtesting

We chose the data under the constraint restriction, and we can see from Figure 15 that the stock return of the year 2000 is below the SPX of 2001. If the CAPM model is feasible, these data should all above the SPX index, but the present result is not better than the expected result, which suggests that the CAPM model may not be realistic, or it may be caused by the insufficient number of stocks.

4. Conclusion

The aim of this article is to optimize a portfolio of 10 stocks by calculating their return values and using two different models - the index model and Markowitz Model theory. The article begins by calculating the return values of 10 stocks, ensuring they follow the Gaussian distribution. Then the

index model and Markowitz Model theory under the CAPM model is applied to compute Excess Returns and Residual Returns. Using these returns, Annual Average Return, Annual Stdev, beta, alpha, and residual Stdev indices was calculated under both models. Adding five constraints to these indices, the corresponding portfolio and sharp ratio was computed. Finally, comparison between the portfolios by drawing corresponding efficient frontier diagrams was conducted. The results have showed that a better performance was made by the Markowitz Model in the aspect of producing a better portfolio optimization compared to the Index Model in all the constraint except constraint four.

However, the Markowitz model has some drawbacks. First, it requires numerous estimates to calculate the covariance matrix. This means that it needs a large amount of dataset in order to generate a portfolio optimization. Second, it doesn't offer a reliable method for predicting stock risk premium. Relying solely on historical returns to predict future returns is a significant flaw. It is important to note that unlike the diversified portfolios and efficient asset allocation emphasized by the Markowitz model, the Index model has systematic risk that cannot be diversified. We use the Index model's correlation coefficient beta to express how much the market moves on average after a change in the investment return of a single stock. The index model can simplify the estimation of the covariance matrix. Additionally, by breaking down systemic risk and firm-specific risk, the estimation of enhanced security risk premium can be optimized. So even though Markowitz Model have been testified to have generate a better optimal portfolio, the complexity of its calculation and large number of data value needed makes it less convenient to be put into practical use than Index Model, so investors can choose the model that best fit their status quo.

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