Comparison of Price Simulation of European Lookback Option and Discrete Asian Option

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Abstract. As a matter of fact, this study focuses on the comparison of European Lookback Option and Discrete Asian option based on Monte Carlo Method and Binary Tree model. In the first step, options and methods are introduced, together with their mathematical principles and expressions. Next, parameters used are provided, including ideal condition and historic data of Unilever. Simulations are made and price gap and price distribution are discussed. Then, sensitive researches are done through changing volatility and strike price. The relationship among price gaps method and the principles of the method has also been taken into consideration. It is found that price trend based on Monte Carlo Method is linear-like while price based on Binary Tree Model is exponential-like. Price gap keeps gradually increasing, which is due to characteristics of the method. Moreover, for a given and fixed volatility, the higher the strike price, the lower the price simulated and vice versa, this is explained mathematically. Finally, limitations, future outlooks and conclusions are made.

Keywords: Monte Carlo Method, Binary Tree Model, Option Pricing, Financial Mathematics.

1. Introduction

It is unfair to say that option process is pretty modern and is recently emerged. History of option can be dated back to ancient Greek civilization. During that time, options were used to offset costs during goods transactions when commerce [1]. Later, commercial activity centration improved the development of option and there appeared the Antwerp exchange. Although Antwerp collapsed in 1585, the form of options passed down. Amsterdam and in London became important places for option trading. Then, a milestone in the history of option appeared during 1634 to 1637, which was also called tulip mania. At that time, even ordinary people began to buy tulip mania, they could earn hundreds and millions of florins in just a few days [2]. This farce ended at the end of 1637, when it was general and common for forward contracts to default.

As to modern option, the establishment of Chicago Board Options Exchange (CBOE) was definitely the beginning of modern option market [3]. In 1973, Fisher Black and Myron Scholes came up with Black- Scholes Formula as an excellent solution for option pricing [4]. With the advance of new century, well developed network and trading system made options trading convenient and it attracted more people to invest. Options nowadays can be roughly divided into Vanilla Option and Exotic Option based on their structures. More specific, based on exercise time there are European Option, American Option and Asian Option. Based on assets related there are Stock Option, Index Option, Fores Option and etc. Based on profiting pattern there are Call Option and Put Option [5]. After the appearance of option, mathematicians began finding methods to price it. Date back to 1900s, Louis Bachelier came up with random walk model in his doctoral thesis [6], which was considered as the starting point of option pricing theory. Later, based on no arbitrage hypotheses, Black- Scholes Formula was used as a powerful method for option pricing. Then, Fischer Black and Myron Scholes improved option pricing model [7].

With the development of stochastic analysis, researchers began pricing option prices via stochastic differential equations (SDEs) based on Itô integral, Black-Scholes Formula can also be seen as a special kind of SDE. Details related to this can be found in [8, 9]. Although the effectiveness and usefulness of SDEs, it is hard for ordinary people without mathematic background to use them while investing. So other mote approachable methods should be used. Thus, there appears Monte Carlo

Method, which can provide numerous random data based on given parameters, which will be discussed later.

2. Models and Method

2.1. Binary Tree Model

Before something about options, two special model should be discussed, which are Binary Tree Model and Black-Scholes Model. As to Binary Tree model, maturity time T is a fixed one and holders can only exercise option at T, 2T…, nT. Price will either go upward or downward with corresponding factors *u* and *d*. If one denotes *S (0)* as the initial price, then at time T, which is the first maturity time, *S(T)* will only take two values, i.e., *S (0) *u* or *S (0) *d*. Risk-neutral assumption is the most important assumption in Binary Tree Option Pricing, risk free and continuously compounded interest rate *r* is introduced to denote the time value of money. After time T, *S (0)* increases to $S(0)* e^{-rT}$. Another assumption is that the following equation always holds [10]:

$$d < e^{-rT} < u \tag{1}$$

If the probability that price go upward is *P*, then *P* can be written as:

$$P=\frac{e^{-r}-d}{u-d} \ and \ \ 1-P=\frac{u-e^{-r}}{u-d} \tag{2}$$

Here, *1-P* means the probability of price go downward. Then the expectation of price at time nT can be written as:

$$E(S(nT)) = S((n-1)T) \times u \times \frac{e^{-r}-d}{u-d} + S((n-1)T) \times d \times \frac{u-e^{-r}}{u-d} \tag{3}$$

By following this equation, one can know the expectation of option at time T. However, this model is simple for it does not take volatility of asset into consideration. If volatility is taken into consideration, *u* and *d* can be denoted by:

$$u = exp\{\left(r-\frac{1}{2}\sigma^2\right)\delta_t + \sigma\sqrt{\delta_t}\} \tag{4}$$

And

$$d = exp\{\left(r-\frac{1}{2}\sigma^2\right)\delta_t - \sigma\sqrt{\delta_t}\} \tag{5}$$

Where *r* is risk free return rate, σ is volatility and δ_t is timestep.

2.2. Monte Carlo Method

In finance researching, it is almost impossible to gain floods of data to do researches such as predicting future prices, assessing risks and so on. It is unrealistic to create a stock market crash in order to test the market conditions after a stock market crisis. Thankfully, Monte Carlo method provides a way out. The key point of Monte Carlo Simulation is generating large number of random samples and using statistical inference to solve problems, especially those do not have analytic solution [11]. Answers of approximated solution, price or distribution can be obtained by analyzing a large number of independent random samples. In this article it is used as a powerful method to simulate path of options. Mathematically speaking, the key idea of Monte Carlo Method is: Let X as

a random variable, f(x) as the probability density function (PDF) of X, g(X) as a function of X and g(X) satisfies:

$$E[g(X)] = \mu, Var[g(X)] = \sigma^2 \tag{6}$$

Then N sample points are taken, thus $g(\bar{X}_N) = \frac{1}{n}\sum_{i=1}^{N} g(X_i)$ and by law of large numbers, for any given $\varepsilon > 0$, there is always an equation:

$$P\left(\left|\lim_{N\to\infty} g(\bar{X}_N) - \mu\right| > \varepsilon\right) \to 0 \tag{7}$$

Let $\hat{\mu}$= $\lim_{N\to\infty} g(\bar{X}_N)$, then $\hat{\mu}$ is called Monte Carlo Simulation of μ. Standard error of convergence is $O(\frac{\sigma}{\sqrt{N}})$, and the convergence order is $O(\frac{1}{\sqrt{N}})$ [12].

3. Introduction of Options

3.1. European Fixed Lookback Option

The definition of European Option is that holders are not allowed to exercise option before maturity time. Exercise price equals to the strike price *K* and *S(T)*. Thus, European Call Option can be denoted as *max(S(T)-K, 0)* and European Put Option is *max(K-S(T), 0)*. The definition of Lookback Options is that holders of Lookback Options are allowed to exercise the option at the highest(lowest) price at the maturity time with respect to strike price *K*. The choice of *K* divided different kinds of Lookback Options. If *K* is a fixed price, then the option is called Fixed Lookback Option. If *K* equals to *S(T)*, which means *K* is a floating price, then the option is called Float Lookback Option. Mathematical formula can be found as following

$$V = max(L - K, 0) \tag{8}$$

Here *L* is asset price and *K* is striking price. As Float Lookback Option are too complicated to be simulated by Monte-Carlo Method while float means, where float indicates that changes in the price during the last period should also be considered, but there may be extreme values in the price of the last period, which may affect the overall simulation value. So, in this article only European Lookback Call Option with fixed price K will be discussed. And as the way to calculate call option and put option is similar, in this article, only Lookback Call Option will be discussed.

3.2. Asian Option in Discrete Condition

Asian Option is different from European Option, it is a path-dependent option. Price during its maturity time influences its price. Merits of Asian Option are that it avoids the possibility of speculators manipulating the underlying asset price to make huge profits when approaching the maturity date and it avoids artificial fluctuations in option prices to some extent [11]. Price of Asian Option can be calculated through arithmetic average and geometric average. The way to calculate these two averages can be found in Eq. (9) and Eq. (10):

Arithmetic Average:

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \tag{9}$$

Geometric Average:

$$\bar{X} = \sqrt[n]{X_1 X_2 \ldots \ldots X_n} \tag{10}$$

For the sake of convenience in calculation, geometric average is calculated as:

$$\bar{X} = \sqrt[n]{X_1 X_2 \dots \dots X_n} = exp\{\frac{1}{n}\sum_{i=1}^{n} ln\ (X_i)\} \tag{11}$$

There are two types of Asian options: discrete and continuous time. As Asian Options with continuous time involves integral calculation and numerical simulation, in this article only Asian Option with discrete time will be discussed. If option price is denoted by V, then it can be calculated by:

Arithmetic Average:

$$J_T = \frac{1}{n}\sum_{i=1}^{n} S_{T(i)} \tag{12}$$

Geometric Average:

$$\begin{aligned} J_T &= \sqrt[n]{S_{T(1)} S_{T(2)} \dots \dots S_{T(n)}} \\ &= exp\{\frac{1}{n}\sum_{i=1}^{n} ln\ (S_{T(i)})\} \end{aligned} \tag{13}$$

As in option pricing, only the result of price is concerned. In other words, researchers and investors care neither the trend or the growth rate nor the price path, they just focus on the value and profit they can get. In this article, arithmetic average is used to get the estimated price, this does not mean that geometric average is useless, it can be used in the simulation and estimation of growth rate or lost rate and other kinds of value that should take the trend of price into consideration. Strike price *K* used in Asian Option are the same as used in Lookback Option, i.e., fixed price *K* or the latest price *S(T)*, they are called Fixed Strike Asian Option and Floating Strike Asian Option correspondingly. And they can be calculated by following formulas:

Fixed:

$$V_T = max\ (J_T - K, 0) \tag{14}$$

Floating:

$$V_T = max\ (J_T - S_T, 0) \tag{15}$$

4. Simulation Results and Sensitive Research

4.1. Parameters

To better simulate options, two kinds of parameters will be used. The first kind of parameter is ideal parameter, which can be found in Table 1. To better approximate reality, here data of Unilever is chosen. Data and parameter come from [13, 14]. Then parameters can be listed in Table 2.

Table 1. Parameter used in ideal condition.

Symbol	Corresponding Meaning	Value
S(0)	Asset Price at Initial Time	100
K	Strike Price	100
r	Risk Free Return Rate	0.02 (2%)
sigma	Volatility	0.20
T	Maturity Time	5
num_simulations	Simulation Times	10000
num_steps	Time Steps (Trading Day per Year)	252

Table 2. Parameter used in for UniLever.

Symbol	Corresponding Meaning	Value
S(0)	Asset Price at Initial Time	39.9
K	Strike Price	39.9
R	Risk Free Return Rate	0.0525 (5.25%)
sigma	Volatility	0.2002
T	Maturity Time	5
num_simulations	Simulation Times	10000
num_steps	Time Steps (Trading Day per Year)	252

4.2. Simulation of European Lookback Option with Discussion

A special parameter used in European Lookback Option is the depth of Binary Tree Model, one denotes sit as n. It can reflect the extend of discretization in model. Larger n means larger discretization and more data, which can help to capture the fluctuation of stock price. However, deeper binary tree means more time and energy to calculate and simulate. Here one sets parameter *n* as 50. By using Binary Tree Model and Monte Carlo Method, simulation result of option prices can be shown in Table 3 and Fig. 1.

Table 3. Simulation Result of European Fixed Lookback Option

European Lookback Option	Ideal Condition	UniLever
Monte Carlo Method	39.02	7.29
Binary Tree Model	21.93	9.21

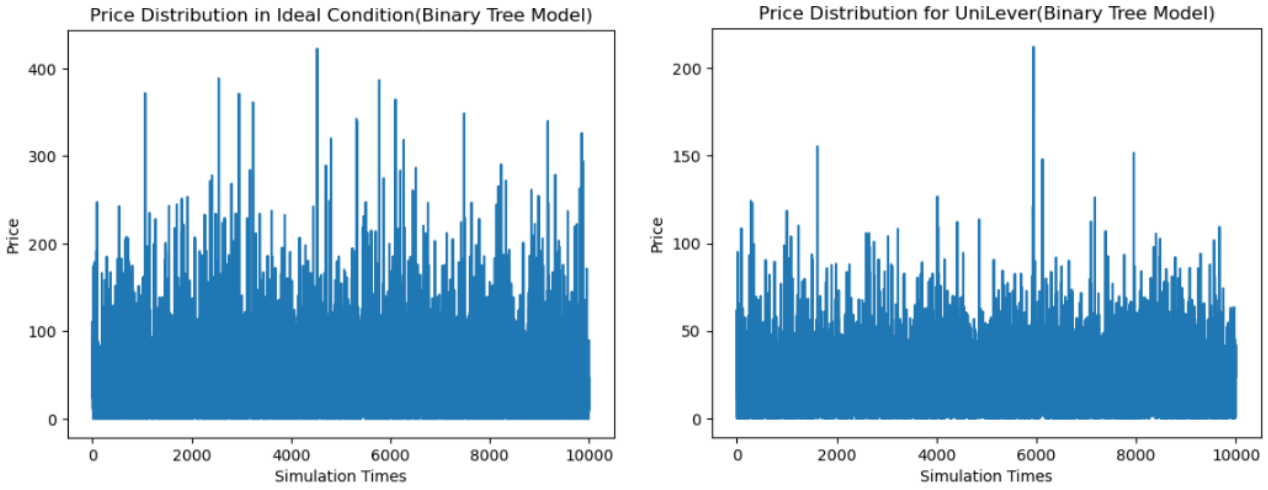


Figure 1. Price Distribution (Photo/Picture credit: Original).

4.3. Simulation of Discrete Asian Option with Discussion

Using Binary Tree Model and Monte Carlo Method, simulation result of option prices can be listed in Table. 4, Fig. 2 and Fig. 3.

Table 4. Simulation Result of Discrete Asian Option

Discrete Asian Option	Ideal Condition	UniLever
Monte Carlo Method	12.53	5.46
Binary Tree Model	22.02	5.52

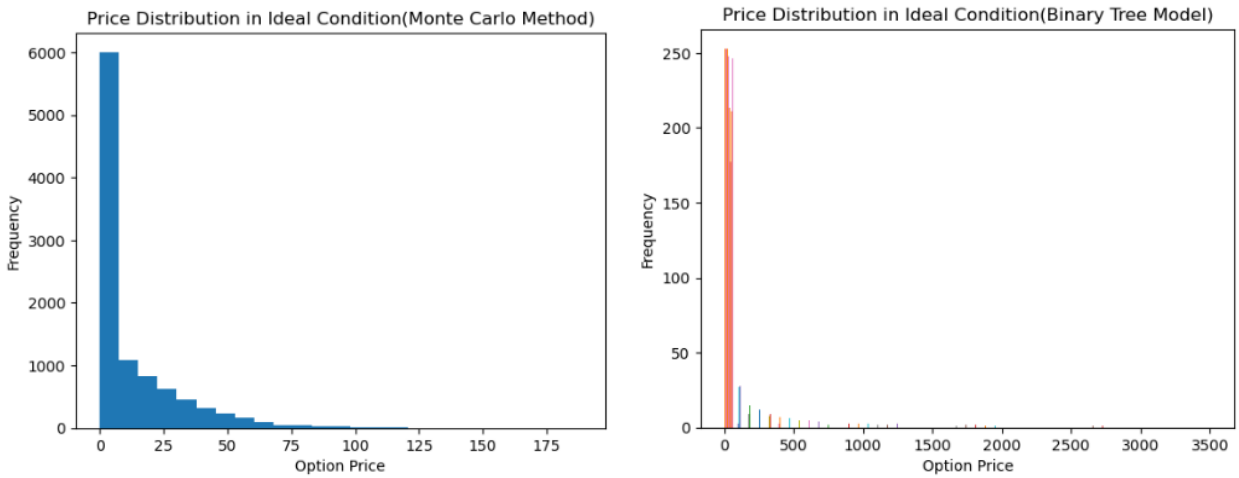


Figure 2. Price Distribution in Ideal Condition (Photo/Picture credit: Original).

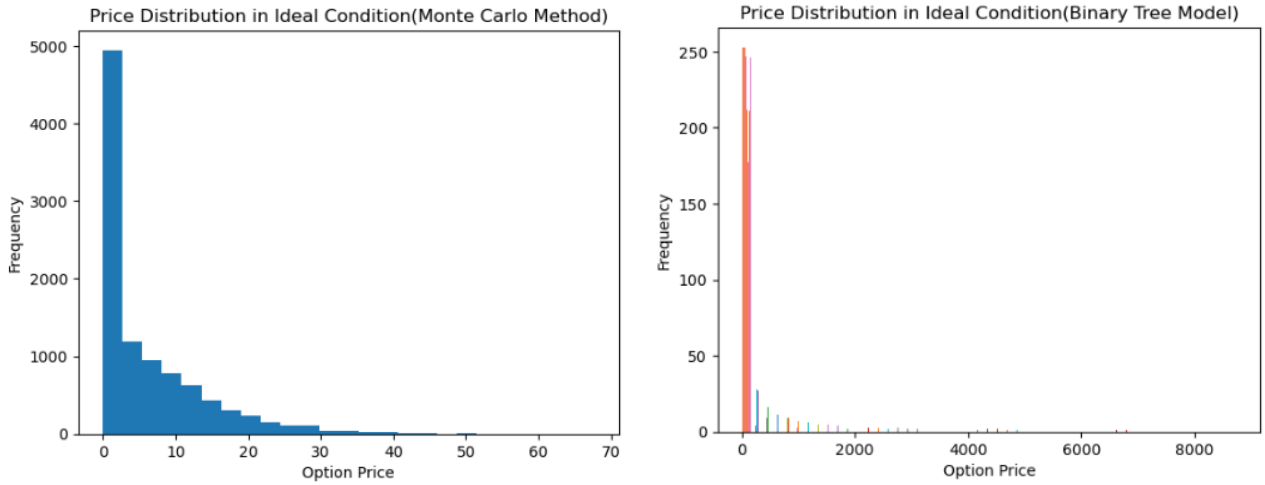


Figure 3. Price Distribution for UniLever (Photo/Picture credit: Original).

4.4. Discussion

One can find that when stock price is relatively small, price gap simulated by two methods tend to shrink, this can also be found through price distribution as shown in Fig. 1. Compared with ideal condition, price simulated for Unilever, high price simulated tend to appear less frequently, in other words, price simulated for Unilever always fluctuates within a small range. What's more, the price distribution based on Monte Carlo Method as shown in Fig. 2 and Fig. 3, Their value distribution is continuous, which means that every number within a certain range may become a price of option while the price distribution based on Binary Tree Model only takes certain value, this is determined by the character of Binary Tree Model. What character these two models share is that the larger the distance of 0, the smaller the probability of its occurrence, which can even be said to decrease exponentially. This can be known from the properties of normal distribution.

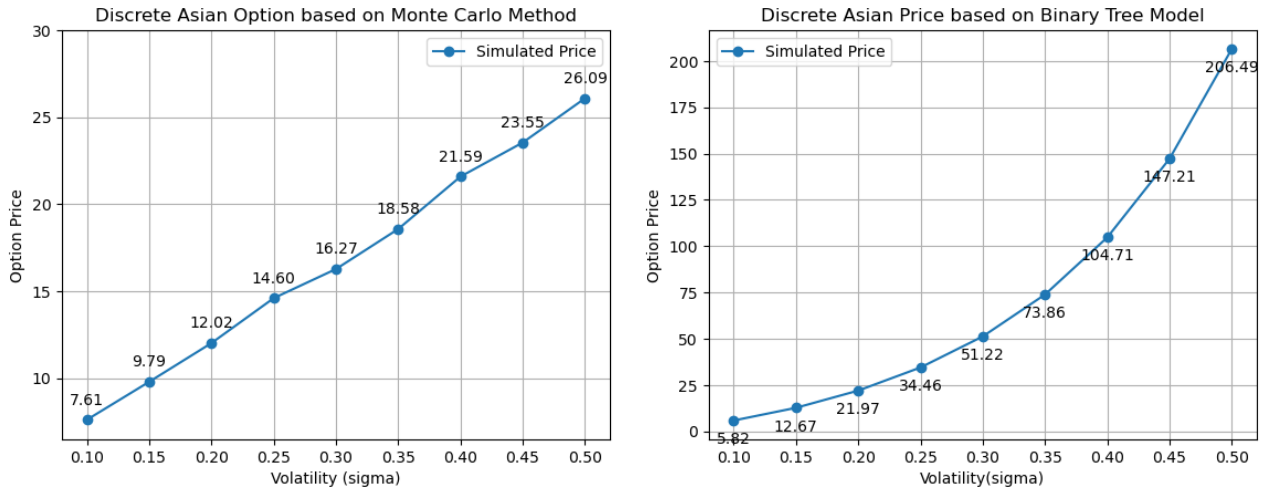


Figure 4. Discrete Asian Option with different volatility (Photo/Picture credit: Original).

4.5. Sensitive Research

During simulation process, volatility used is a constant, which is get through measurement of historical data. But in real life, volatility is influenced by numerous factors and changes rapidly. The key point is that how to measure the effect of volatility toward profit and price, in other words, how significant is the impact of volatility on prices. Thus, in this section, prices based on different values of volatility and reasons toward potential trends are given. As every coin has two slides, bigger volatility means more chance for the asset price to fluctuate. Holders have the opportunity to earn higher returns, but they may also suffer more losses. Letting sigma changes via values in number list [0.1, 0.15, 0.2, 0.25, 0.3, 0.35, 0.4, 0.45 ,0.5], results can be illustrated in Fig. 4.

In terms of the two subplots of figure 5, although both show an upward trend, the price trend based on Monte Carlo Method tends to be more linear-like, while the price trend based on Binary Tree Model resembles more of an exponential function. Additionally, price change of Monte Carlo Method is relatively small, while price change obtained by Binary Tree Model are relatively large. This can be explained through the construction and mathematical theorem of these two models. Although Binary Tree model has a relatively small computational load, results it output are fixed and determined, which may set obstacles when simulating classic Asian Option as the price of an Asian option is based on the average value of the asset price. Researchers have found that it can be used into European-style Asian Option [15]. Besides, Binary Tree Model assumes that at each step that the asset price can only go up or down, this can also yield difference. Thus, different volatility levels may lead to different kinds of returns, which is determined by the method used and by the option chosen.

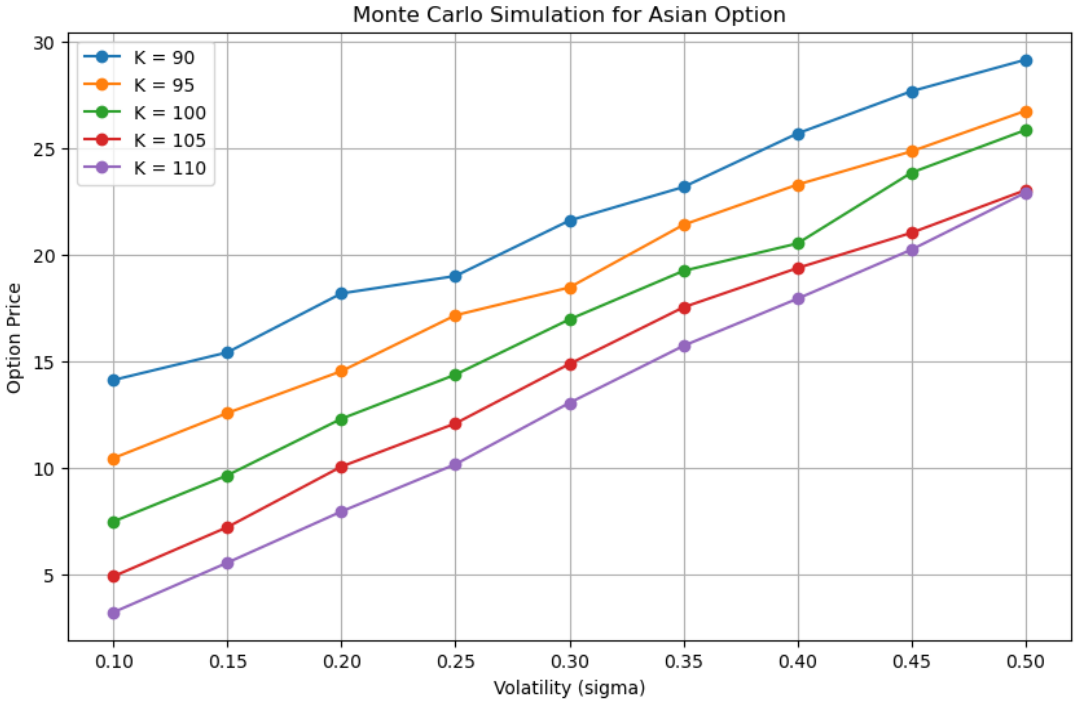


Figure 5. Discrete Asian Option with different volatility (Photo/Picture credit: Original).

Then, based on the volatility set, letting strike price K changes via values in number list [0.1,0.15,0.2,0.25, 0.3,0.35, 0.4,0.45,0.5] and as depicted in Fig. 5, prices with different strike price K keeps increasing with some fluctuation. And line with higher strike price K tends to be lower in the whole paragraph, this makes sense for option price is the larger number between zero and the result of asset price minus strike price K. Thus, if K is larger, which means option price tends to be smaller (but still larger or at least equal to zero). This way can also be applied into Binary Tree Model.

5. Limitations and Future Outlooks

Limitations of this article are that it only focuses on two simple options and fails to expand its range, there exists vast kinds of options throughout the market such as Snowball Option and Currency Option. What's more, due to time and page limitation, this article fails to introduce more advanced method and theorem to research option pricing and only Monte-Carlo Method and Binary Tree model are used. Given the fact that Black- Scholes Formula, which is famous and well-accepted in option pricing, is only active and useful to traditional European Option. There remain numerous methods for option pricing such as SDEs and numeral method.

As to future outlook of option pricing, there are several potential innovation points. The first one is the pricing of cross-asset options, such as currency and goods combined option. There are floods of essays focusing on portfolio management (e.g., [16]) while only few essays about cross-asset options. Whether effective portfolio management can be applied to option, especially cross-asset option, in investment remains a question. What's more, the improvement of numerical methods and the increase in computational power may lead to faster calculation and less time required, which can contribute to high-frequency options trading. And with the gradually maturity of financial market, the possibility for much more complicated option to emerge is high, which can much deeply reflect the structure of the financial market.

6. Conclusion

To sum up, this study focuses on the comparison of option price prediction. Two kinds of option are simulated (European Lookback Option and Discrete Asian Option) via two methods (Monte Carlo Method and Binary Tree Model). In this article, options and methods are introduced then simulations

are done. Discussions about price simulated are done. It is found that initial stock price can influence price gap and price distribution based on these two methods are different due to assumptions. The one closely following is sensitive researches about volatility and strike price. It is found that there exists price gap between two methods based on the same parameter and explanations are given. A linear-like trend can be found with different volatility and strike price. For a fixed volatility parameter, smaller strike price can yield higher option price, which can be explained mathematically. However, this article fails to expand its range and introduce more advanced method and theorem due to various limitations. It is expected that cross-asset options, knowledge from portfolio management and the improvement of numerical methods can yield more complicated and much deeply option that can reflect the structure of the financial market.

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