Portfolio Analysis of Five Stocks Under Different Constraints Based on Markowitz Model & Index Model

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Abstract. Since the dawn of the 21st century, while there has been an abundance of talented fund managers, truly exceptional asset portfolios have been rare. Addressing this phenomenon, the study delves into the current state of asset management, employing two renowned financial models: the Markowitz model and the Index model. These models are utilized to analyze the returns and risks of five high-performing assets. Additionally, acknowledging the diversity of investor preferences, this paper incorporates five different constraints to cater to a wider spectrum of client needs. The crux of the research is the optimization of the asset portfolio, grounded in these models and constraints. The paper findings indicate that the integration of the Markowitz and Index models with the capital market line and the efficient frontier proves highly effective in the management of risk asset portfolios. In this paper, the Markowitz model consistently excels over the index model in both minvariance and maxsharpe scenarios. This study has important reference value for investors in related fields.

Keywords: Portfolio optimization, Portfolio, Markowitz model, Constraint.

1. Introduction

The construction of an investment portfolio is a critical endeavor in the field of finance, integral to achieving long-term investment goals. It involves the strategic combination of various assets to maximize returns while minimizing risk-mean-Variance by American economist Markowitz [1]. A well-constructed portfolio not only aligns with the investor's risk tolerance and investment horizon but also capitalizes on diversification to reduce unsystematic risk, thereby enhancing the potential for stable and favorable outcomes.

After a careful study of the portfolio of financial risk assets, this paper notes that most of the current industry research focuses on the theory itself and the further optimization of the portfolio. Bai discusses a recognized issue in the application of the Markowitz mean-variance optimization: the estimated returns often significantly deviate from theoretical optimal returns. To address this, Bai has developed a new bootstrap method to correct the estimation of optimal returns and asset allocation [2]. João, Guilherme, and André put forward the utilization of a broad category of dynamic factor models for achieving optimal bond portfolio configurations, as well as for crafting a duration-constrained mean-variance optimization to enhance bond indexing [3]. Qi conducted a study on portfolio selection focusing specifically on enterprises within the traditional Chinese medicine sector [4]. Whats more, Wang analyzed a selection of eight stocks from the artificial intelligence sector. By applying Markowitz's portfolio theory, Wang constructed the efficient frontier and identified the optimal portfolio, taking into account the varying degrees of risk aversion among investors and aligning the selection with their investment utility [5]. Additionally, Naccarato et al. focused their analysis by choosing the 30 most highly capitalized stocks, exclusively within the European financial sector, ensuring a pure financial perspective by omitting stocks from other sectors [6]. While Alla Petukhina also used Projected Gradient Descent (PGD) techniques and modern robust statistical tools to stabilize the costs and weights of global minimum variance Markowitz portfolios and improve their out-of-sample performance [7]. On a different note, Zhi et al. integrated advanced Copula-based methods to refine portfolio optimization strategies for inventory financing, concentrating on the nuanced dynamics of monthly collateral returns from seven principal commodities in the metal industry [8].

Building upon the insights gleaned from previous paragraphs, this study meticulously selects the equities of five exemplary corporations for asset allocation. The analytical process encompasses two decades' worth of data on these stocks, culminating in the extraction of key financial indicators: Annual Average Return, Annual Standard Deviation, Beta, Alpha, and Residual Standard Deviation. Additionally, by assessing the interrelations between each pair of stocks, a comprehensive correlation matrix is constructed. This matrix serves as a pivotal tool for risk management within the portfolio, utilizing the interplay of positive, negative, or negligible correlations among assets to temper volatility.

The research further investigates the impact of strategic constraints on portfolio performance. A noteworthy finding is that imposing a restriction to exclude the SPX index from the asset mix enhances the portfolio's yield and risk profile, particularly when contrasted with a scenario devoid of any constraints. This result underscores the potential benefits of selective constraint application in portfolio optimization, suggesting that strategic exclusions can lead to more favorable risk-return trade-offs.

2. Data

Yahoo Finance and Investing are renowned for their exceptional accuracy in providing industry information, and their data is known for its high reliability. For this research, data was primarily sourced from Yahoo Finance (https://finance.yahoo.com/) and corroborated with information from Investing (https://cn.investing.com/equities/). To ensure the robustness of the data used in this study, the time frame selected spans from November 5, 2001, to December 5, 2021. Furthermore, to facilitate risk allocation and hedging in this portfolio, the selected stocks are diversified across various sectors, ensuring a broad coverage of the market landscape. This paper converts the data into (AAR) and Annualized Standard Deviation (ASD)as well as the beta, alpha coefficient and annualized residual standard deviation (ARSD), with key details presented in Table 1.

Table 1. Descriptive statistics of the selected assets

	SPX	ALK	BAC	AKAM	ALL	AMZN
AAR	7.542%	17.431%	11.101%	28.136%	10.080%	33.796%
ASD	14.850%	37.734%	39.338%	63.121%	24.884%	41.410%
β	100.000%	117.790%	159.545%	165.497%	105.624%	135.128%
α	0.000%	8.547%	-0.933%	15.653%	2.113%	23.604%
ARSD	0.000%	33.435%	31.403%	58.140%	19.317%	36.223%

As can be seen from Table 1, AKAM and AMZN present the highest returns but also come with high volatility and market risk. ALK and BAC are highly sensitive to market movements with their significant betas. ALL appears to have the most balanced profile with the lowest return but also lower risk and a positive alpha, implying a more stable performance compared to the benchmark.

3. Method

The core idea of Harry Markowitz's influential work in financial economics, particularly in portfolio theory and asset allocation, revolves around the concept of rational investor behavior and an intrinsic aversion to risk [9-10]. Markowitz postulates that investors, as rational actors, are inherently reluctant to undertake additional risk without the promise of proportionally higher expected returns. This principle underscores the critical balance between risk and return in investment decisions. The crux of investment in securities and other high-risk assets lies in addressing two pivotal factors: the anticipated return and the inherent risk associated with these investments. Consequently, a pressing challenge for market investors is to accurately measure and effectively balance these two indicators for optimal asset allocation. Markowitz's pioneering mean-variance portfolio optimization model, which forms the bedrock of contemporary portfolio optimization, is an

extension of this fundamental approach. Despite the evolution of various theories and models in asset allocation over time, Markowitz's theory remains the foundational cornerstone, often regarded as the "ancestor" of modern asset allocation strategies.

Building on the significance of Harry Markowitz's model and the capital asset pricing model (CAPM), which is an extension of the index model, the implementation of these theories in portfolio investment is a logical progression. These models form the analytical framework for constructing a diversified investment portfolio, guiding investors on how to allocate assets to maximize expected returns for a given level of risk or, conversely, to minimize risk for a given level of expected returns.Therefore, this paper adopts the Markowitz portfolio and the index model portfolio to simulate the SPX and five stocks in various industries.

In this paper, the expected return formula and the profit effective frontier formula of Markowitz model are as follows:

$$r_M = \sum_i r_i * w_i \tag{1}$$

$$Var = \vec{w}^T \Sigma \vec{w} \tag{2}$$

The equation (1) represents the expected returns the return of Markowitz model; r_i is the annualized average return of stock i.

The equation (2) calculates the variance of the portfolio, which is a measure of the portfolio risk. Here, $\vec{w}$ is a column vector of asset weights, Σ is the covariance matrix of asset returns, and $\vec{w}^T$ is the transpose of $\vec{w}$.

The above formula is the core formula of the Markowitz model used to calculate the portfolio return and return risk, followed by the portfolio expected return formula of the index model and the portfolio risk (standard deviation) formula:

$$E(R) = \sum(w_i * \beta_i) * R_M + \sum(w_i * \alpha_i) \tag{3}$$

$$\sigma_P = \sqrt{(\sigma_M \beta_P)^2 + \sum_{i=1}^{n} w_i^2 \sigma^2(\varepsilon_i)} \tag{4}$$

The equation (3) is the expected return of the portfolio:

w_i: Weight of the i-th asset in the portfolio, representing the proportion of the total value of the portfolio allocated to this asset.

β_i: Beta of the i-th asset, a measure of the asset's volatility in relation to the market. A beta greater than 1 indicates that the asset is more volatile than the market, whereas a beta less than 1 indicates that the asset is less volatile.

R_M: Expected market return, typically represented by the return of amarket index like the S&P 500.

α_i: Alpha of the i-th asset, representing the expected return in excess of the return predicted by the market as a whole. It reflects the individual asset's performance independent from the market movement.

The equation (4) calculates the total risk (standard deviationσ_P) of the portfolio, encompassing both systematic and unsystematic risks.

σ_M represents the standard deviation of the market, indicating market risk.

β_P represents the sensitivity of the portfolio relative to the market, also known as the market risk coefficient.

w_i is the weight of the i-th asset in the portfolio.

σ^2 is the variance of the idiosyncratic risk (unsystematic risk) of the i-th asset.

4. Result

By calculating and observing the correlation coefficient matrix between the S&P index and these five stocks, it can be observed that because Allstate Corporation and Bank of America Corporation are both in the Financial Services industry, there is a strong correlation between the two companies,

and their stock movements are highly synchronized with the fluctuations of the S&P index. In contrast, Akamai Technologies, Inc., which represents the technology industry. The correlation with any other stock is less than 0.3, indicating that its price moves relatively independently of the rest of the group. As a result, investors looking to reduce volatility in their portfolios may want to consider adding Alaska Air Group, Inc. For portfolios where the objective is to minimize overall volatility, a strategy favoring stocks with lower correlations to SPX, such as ALK and AKAM, and those that manifest minimal inter-stock correlations, akin to the relationship between AKAM and other stocks, would be advisable (See Table 2).

Table 2. Correlation coefficient between five stocks

	SPX	ALK	BAC	AKAM	ALL	AMZN
SPX	1.000	0.464	0.602	0.389	0.630	0.485
ALK	0.464	1.000	0.275	0.186	0.234	0.182
BAC	0.602	0.275	1.000	0.061	0.547	0.139
AKAM	0.389	0.186	0.061	1.000	0.200	0.271
ALL	0.630	0.234	0.547	0.200	1.000	0.198
AMZN	0.485	0.182	0.139	0.271	0.198	1.000

In real life, there are various types of investors, each with their own unique considerations when it comes to investing. These considerations are reflected in the construction of investment models, which incorporate different constraints. Among the most common constraints are the following five:

Constraint 1 (Sum of absolute weights ≤ 2): In practice,regulatory frameworks such as FINRA's Regulation T necessitate that clients maintain certain capital levels against their positions. This constraint ensures that the portfolio aligns with such regulations, with the sum of absolute values of weights not exceedingly twice the capital, effectively simulating a leveraged position where the investor's own equity funds at least 50% of the investments.

Constraint 2 (Absolute weight of each asset ≤ 1): Reflecting the individuality of investors and fund managers, this constraint provides a "box" within which each asset's weight must fall, allowing for personalized investment styles and risk preferences to dictate portfolio composition without overexposure to any single asset.

Constraint 3 (No constraints): This setting is akin to a blank canvas, providing an opportunity for each asset's inherent characteristics to be fully expressed within the portfolio. It reveals the unconstrained potential and performance of assets, offering a baseline against which the impact of other constraints can be measured.

Constraint 4 (No short positions, all weights non-negative): Mirroring restrictions in markets where short-selling is prohibited due to less mature financial systems, this constraint ensures that all positions are long-only. It is particularly relevant for portfolios tailored to markets like China's A-share market, promoting a conservative investment approach in accordance with local regulations.

Constraint 5 (Weight of the first asset, presumably a broad market index, is zero): Often, indices like the SPX are included in portfolios as a hedge against volatility. By explicitly excluding such an index (setting its weight to zero), this constraint seeks to assess the portfolio's performance in its absence, exploring whether foregoing this conventional risk management tool can lead to superior returns or a more favorable risk profile.

The following is the efficient boundary of portfolio investment and portfolio performance based on the Markowitz model and the exponential model under five constraints (See Fig. 1, Fig. 2, Table 3, and Table 4):

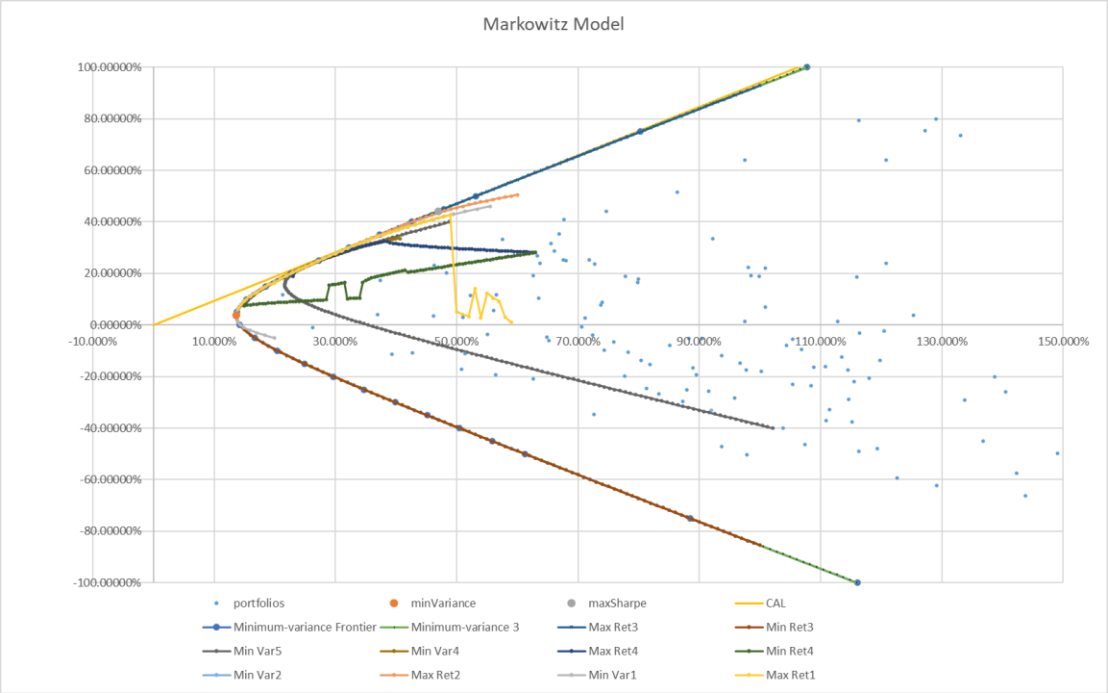


Fig.1 Effective frontiers under different constraints for mean-variance model

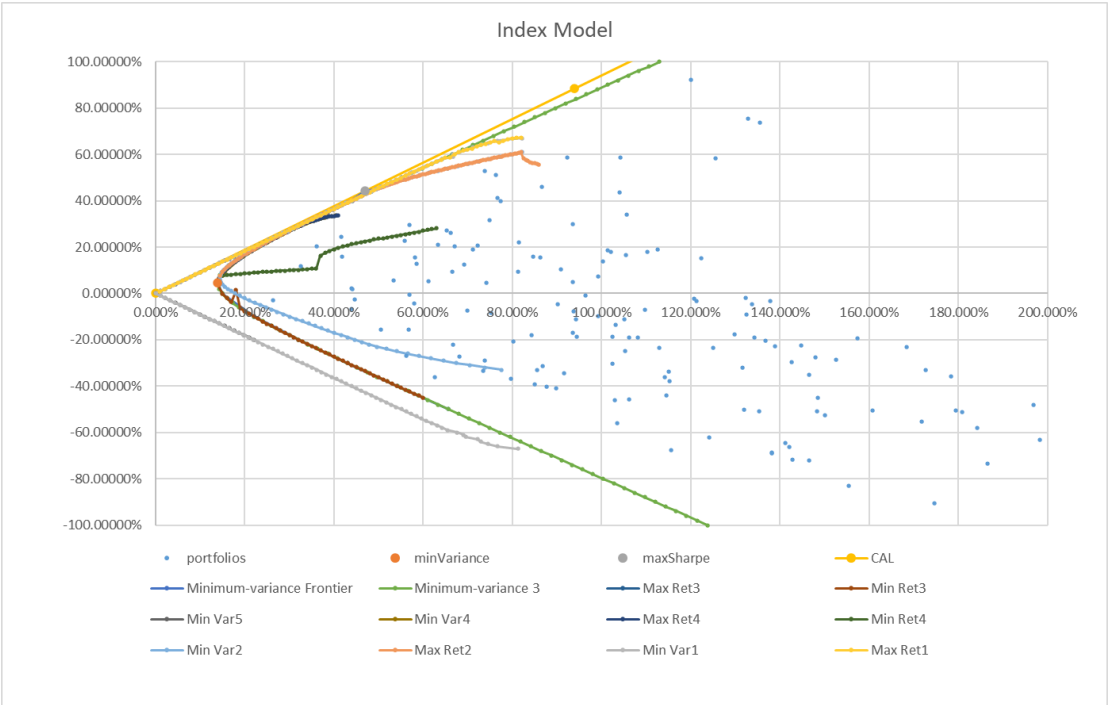


Fig. 2 Effective frontiers under different constraints for index model

Table 3. Portfolio performance of Markowitz model

Markowitz Model									
constraint 1	SPX	ALK	BAC	AKAM	ALL	AMZN	Return	StDev	Sharpe
minVariance	130.710%	-3.410%	-15.350%	-5.080%	0.330%	-7.210%	3.729%	13.644%	27.332%
maxSharpe	-50.000%	29.860%	7.360%	12.400%	35.030%	65.350%	31.357%	33.617%	93.275%
Markowitz Model									
constraint 2	SPX	ALK	BAC	AKAM	ALL	AMZN	Return	StDev	Sharpe
minVariance	100.000%	1.930%	-11.410%	-3.440%	14.700%	-1.790%	6.524%	14.205%	45.926%
maxSharpe	-100.000%	38.910%	13.690%	16.620%	48.530%	82.250%	38.125%	40.578%	93.953%
Markowitz Model									
constraint 3	SPX	ALK	BAC	AKAM	ALL	AMZN	Return	StDev	Sharpe
minVariance	130.715%	-3.405%	-15.350%	-5.077%	0.328%	-7.210%	3.729%	13.644%	27.332%
maxSharpe	-143.536%	46.825%	19.183%	20.404%	59.597%	97.527%	44.174%	46.959%	94.070%
Markowitz Model									
constraint 4	SPX	ALK	BAC	AKAM	ALL	AMZN	Return	StDev	Sharpe
minVariance	100.000%	0.000%	0.000%	0.000%	0.000%	0.000%	7.542%	14.850%	50.790%
maxSharpe	0.000%	20.860%	1.030%	8.420%	19.950%	49.740%	24.942%	27.368%	91.136%
Markowitz Model									
constraint 5	SPX	ALK	BAC	AKAM	ALL	AMZN	Return	StDev	Sharpe
minVariance	0.000%	19.300%	1.430%	1.900%	61.490%	15.880%	15.623%	21.659%	72.128%
maxSharpe	0.000%	9.200%	0.450%	3.710%	8.800%	21.940%	11.000%	12.070%	91.136%

Table 4. Portfolio performance of Index model

Index Model									
constraint 1	SPX	ALK	BAC	AKAM	ALL	AMZN	Return	StDev	Sharpe
minVariance	-58.473%	-7.065%	-10.624%	-3.542%	-15.717%	104.577%	25.941%	38.630%	67.153%
maxSharpe	-38.388%	26.086%	-3.227%	15.800%	19.323%	61.380%	28.431%	31.145%	91.285%
Index Model									
constraint 2	SPX	ALK	BAC	AKAM	ALL	AMZN	Return	StDev	Sharpe
minVariance	100.000%	1.219%	-7.408%	-2.527%	10.421%	-1.705%	6.696%	14.408%	46.472%
maxSharpe	-47.409%	32.216%	-3.985%	19.512%	23.864%	75.802%	35.111%	38.463%	91.285%
Index Model									
constraint 3	SPX	ALK	BAC	AKAM	ALL	AMZN	Return	StDev	Sharpe
minVariance	130.715%	-3.405%	-15.350%	-5.077%	0.328%	-7.210%	3.729%	13.644%	27.332%
maxSharpe	-6.660%	-2.216%	0.000%	86.221%	0.199%	104.577%	58.734%	74.937%	78.378%
Index Model									
constraint 4	SPX	ALK	BAC	AKAM	ALL	AMZN	Return	StDev	Sharpe
minVariance	100.000%	0.000%	0.000%	0.000%	0.000%	0.000%	7.542%	14.850%	50.790%
maxSharpe	0.000%	21.657%	0.000%	13.700%	7.996%	56.647%	27.580%	30.503%	90.419%
Index Model									
constraint 5	SPX	ALK	BAC	AKAM	ALL	AMZN	Return	StDev	Sharpe
minVariance	0.000%	17.359%	8.747%	2.097%	60.419%	11.378%	14.523%	22.319%	65.068%
maxSharpe	0.000%	23.225%	-7.853%	14.490%	11.364%	58.775%	28.263%	31.136%	90.772%

Attributable to its lower Annualized Average Return and Standard Deviation, SPX commands a substantial allocation in both the minimum variance (minvar) and maximum Sharpe ratio (maxsharpe) portfolios under the Markowitz and index models. This allocation is particularly noteworthy under Constraint 2 (Absolute weight of each asset <= 1) and Constraint 3 (No constraints). Furthermore, in the pursuit of maxsharpe, Amazon's stock consistently holds a considerably larger weight compared to other individual stocks, regardless of whether it's in the Markowitz or index model.

From a broader model perspective, the investment portfolio crafted using the Markowitz model slightly outperforms the index model in both minvariance and maxsharpe scenarios. This superiority is especially pronounced under Constraint 1, where the Markowitz portfolio demonstrates significantly enhanced risk control in minvar compared to the index model. Additionally, the impact of constraints on the Markowitz portfolio is less pronounced in both minvar and maxsharpe strategies. Contrastingly, the unconstrained index model in maxsharpe notably lags behind its constrained counterparts under Constraints 1, 2, 4, and 5.

5. Conclusions

The current landscape of portfolio research predominantly revolves around analyses of broad market dynamics or discrete industrial sectors. This study seeks to expand this scope by conducting a comprehensive portfolio analysis across diverse sectors including technology, financial services, industrials, energy, and consumer cyclicals. Within this study, the Markowitz portfolio theory and the Sharpe Single-Index Model are harnessed to derive the optimal stock portfolio composition, wherein the inherent pros and cons become evident throughout the examination. To mirror the complexity of real-world financial markets, the study integrates five pertinent constraints into these models. This allows for a nuanced exploration of how practical market factors influence portfolio outcomes. Additionally, the research delves into the performance of differing models when subject to identical constraints, as well as the impact of varied constraints upon a single model.This study found that SPX, due to its lower Annualized Average Return and Standard Deviation, secures a significant position in both minvar and maxsharpe portfolios within the Markowitz and index models, especially under certain constraints. Amazon stock is prominent for its higher allocation in maxsharpe across both models. Overall, the Markowitz model outshines the index model in minvariance and maxsharpe, particularly in risk control under Constraint 1. The influence of constraints is less significant on the Markowitz model, whereas the unconstrained index model falls short in maxsharpe.

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