Feynman Integral Problem Solving Techniques: Application on Parametric Integrals

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Abstract. This paper focuses on parametric integration in Feynman integral solving technique. This paper mainly introduces the theory of integration and the method of solving the problem. This paper mainly introduces the methods and techniques of indefinite integral method and definite integral method, and introduces the methods under these two categories in detail. Different methods and techniques are adopted in this paper, and many application examples are listed. Parametric integrals have many advantages, such as rich expressions. Parametric integrals can represent a wide range of functions, including important functions in theory and practice, and have important applications in many fields. Secondly, complex integrals can be solved. For some complex integral problems, especially when the original function is not an elementary function, it may be very difficult to solve them directly. However, parametric integrals provide a new way to solve such problems by introducing parameters and exchanging operation order. Moreover, it can effectively solve practical application problems and promote the development of mathematics. The study of parametric integral not only enricheth the content of mathematical theory, but also enables mathematicians to have a deeper understanding of the properties of functions, the nature of integrals and the internal relations between them through the study of parametric integral.

Keywords: Integral; Calculus; Parametric integration; Feynman integral technique.

1. Introduction

The calculation of definite integral is a core concept in calculus, and the definite integral has clear geometric meaning at first. It represents the area of a curved trapezoid or, more generally, the area of a figure enclosed by a function curve, the X-axis, and two straight lines perpendicular to the X-axis. This geometric intuition not only helps to understand the concept of definite integral, but also makes definite integral play an important role in solving geometric problems [1]. Second, with definite integrals, one can convert instantaneous physical quantities into average or total effects over a period of time, which is crucial for understanding and predicting physical phenomena. In physics, with definite integrals, one can convert instantaneous physical quantities into an average or total effect over a period of time, which is essential for understanding and predicting physical phenomena [2]. In economics, a definite integral can be used to calculate the cumulative value of economic indicators such as costs, benefits, and profits. These applications make definite integrals a powerful tool for solving practical problems. To sum up, the calculation of definite integral is important because it not only has a clear geometric meaning and a wide range of physical, engineering and economic applications, but also has a variety of effective calculation methods in practical applications [3].

In the calculation of definite integral, there are many methods to choose from, and when it comes to parametric integral, the application of these methods will become more complex but more flexible, especially for parametric integral, these integration methods are: direct integration, integration by substitution, integration by parts and so on. The solution of parametric integrals is often more complicated than that of ordinary definite integrals, which requires comprehensive use of many methods and techniques [4].

In this paper, in this paper, section 2 mainly introduces the method and theory of definite integral. Among these two categories, the examples listed in Section 3 mainly use several integral methods such as basic integral formula method, integral by substitution method, and integral by parts method, and mark them in the calculation steps. Namely, it mainly lists many examples to show how to solve the problem of definite integral.

2. Method and Theory

2.1. Integral Theorems

There are many theorems concerning integrals. The first are the integrability Theorems. One is said that if a function is continuous on a closed interval, then the function is integrable on that interval. The other states that if a function is bounded on a closed interval and has only a finite number of breaks, then the function is integrable on that interval. These theorems give sufficient conditions for the integrability of functions and are prerequisites for integral operations [5]. The second is the Newton-Leibniz formula. If a function is continuous over a closed interval and there is an original function, then the definite integral of that function over that interval is equal to the difference between the values of its original function at the endpoints of the interval [6]. This is the basic formula for calculating definite integrals. Finally, there is also the mean value theorem of integrals. If a function is continuous over a closed interval, then there is at least one point in the interval such that the value of the function at that point multiplied by the length of the interval is equal to the definite integral of the function over the interval. This theorem reveals the relationship between the definite integral and the value of a function at a certain point [7].

2.2. Integral Method

In this subsection, the author will firstly introduce the indefinite integral method. The first is the Basic integral formula. For some basic functions, such as power function, exponential function, logarithmic function, trigonometric function, etc., the corresponding indefinite integral formula is listed for direct calculation [8]. The second is the integration by substitution. By introducing new variables to replace some or all of the variables in the original integral, the original integral into a more easily solved form. This method is especially suitable for the integration of complex functions. The third is integration by parts. For the integral of a product function, it can be converted into the difference of the integral of two simpler functions through the integration by parts formula. This method is very useful when solving integrals of complex product functions [9].

To proceed on, there are also some definite integral methods. For the direct integration method, it states that for a function whose original function is known, the definite integral over a given interval is calculated directly using the Newton-Leibniz formula [10]. For the partition integration method, it means that for complex functions or functions that are difficult to integrate directly, it can be divided into multiple subintervals that are easy to integrate on a given interval, and then the definite integral on each subinterval is calculated separately and summed [11]. Finally, there are also pplication of the method of substitution and integration by parts in definite integrals: Similar to indefinite integrals, the method of substitution and integration by parts can also be used in the calculation of definite integrals. Especially when the integrand is difficult to integrate directly over a given interval, these methods can play an important role.

3. Examples and Applications

3.1. Example 1

The first integral is

$$I_{x,z} = \int_0^\infty \frac{\sin(xt)\, e^{-zt}}{t} dt\,. \tag{1}$$

To calculate the integral in Eq. (1), firstly, one takes the partial derivative of x, and one can get

$$\partial_x I_{x,z} = \int_0^\infty \cos(xt)\, e^{-zt} dt. \tag{2}$$

The above formula is a definite integral with respect to the parameters x and z, then the integration by parts method is adopted. Thereore,

$$\partial_x I_{x,z} = -\frac{1}{z}e^{-zt}\cos(xt)\Big|_0^\infty - \frac{x}{z}\int_0^\infty \sin(xt)\,e^{-zt}dt = \frac{1}{z} + \frac{x}{z^2}\int_0^\infty \sin(xt)\,de^{-zt} \tag{3}$$

and also

$$\partial_x I_{x,z} = \frac{1}{z} + \frac{x}{z^2}\sin(xt)\,e^{-zt}\Big|_0^\infty - \frac{x^2}{z^2}\int_0^\infty \cos(xt)\,e^{-zt}dt. \tag{4}$$

The last integral of the above equation can be solved by using the stepwise integral method. By using $\int_0^\infty \cos(xt)\,e^{-zt}dt = \frac{z}{x^2+z^2}$ and since integration and derivation have reciprocal relations, integrate the above formula

$$I_{x,z} = \int_0^x \partial_x I_{x,z}\,dx = \frac{1}{z}\int_0^x \frac{dx}{\left(\frac{x}{z}\right)^2+1} = arc\tan\frac{x}{z}\Big|_0^x = arc\tan\frac{x}{z} \tag{5}$$

Thus, the value in Eq. (1) is $\arctan\frac{x}{z}$.

3.2. Example 2

The second integral is

$$I = \int_0^{+\infty} \frac{sin^2 x}{x^2}\,dx\,. \tag{6}$$

To calculate the integral in Eq. (5), one can firstly introduce a parameter integral which is given by

$$\mathrm{I(t)} = \int_0^{+\infty} \frac{\sin^2(tx)}{x^2}\,dx\ . \tag{7}$$

Obviously, it is found that $I(0) = 0$. In addition, its derivative is

$$\mathrm{I'(t)} = \frac{\partial}{\partial t}\int_0^{+\infty} \frac{\sin^2(tx)}{x^2}\,dx = \int_0^{+\infty} \frac{\sin(2tx)}{x}\,dx = \int_0^{+\infty} \frac{\sin(u)}{u}\,du \tag{8}$$

To proceed, the author should calculate the last integral $\int_0^{+\infty}\frac{\sin(u)}{u}\,du$, which is the famous Dirichlet integral

$$\int_0^{+\infty} \frac{\sin(u)}{u}\,du = \frac{\pi}{2}. \tag{9}$$

Therefore, the expression for $I(t)$ is

$$I(t) = \int_0^t \mathrm{I'(t)}\,dt + I(0) = \frac{\pi}{2}t \tag{10}$$

Thus, the value in Eq. (6) is $I = I(1) = \frac{\pi}{2}$.

3.3. Example 3

The third integral is

$$I = \int_0^\infty \frac{arctan\,x}{x(x^2+1)}\,dx\,. \tag{11}$$

To calculate the integral in Eq. (11), since this integral does not contain the unknown parameters as Example 1, hence one can introduce the parameter t into the arctangent function given by

$$I(t) = \int_0^{\infty} \frac{arc\tan(tx)}{x(x^2+1)} dx \tag{12}$$

Obviously, it is found that $I(0) = 0$. In addition, take the derivative of t to get the exact value

$$I'(t) = \int_0^{\infty} \frac{dx}{(t^2x^2+1)(x^2+1)} = \frac{t}{t^2-1}\frac{\pi}{2} - \frac{1}{t^2-1}\frac{\pi}{2} = \frac{\pi}{2}\frac{1}{t+1}. \tag{13}$$

The second step above uses the rational fraction disassembly method. I(t) can be obtained by integrating the above expression

$$I(t) = \int_0^t I'(t)\, dt + I(0) = \frac{\pi}{2}\int_0^t \frac{dt}{t+1} \tag{14}$$

Finally, $t = 1$ gives the solution to the original integral. Since

$$I = I(1) = \frac{\pi}{2} ln2 \tag{15}$$

Thus, the value in Eq. (11) is $I = I(1) = \frac{\pi}{2} ln2$.

3.4. Example 4

The fourth integral is

$$I = \int_{-\infty}^{+\infty} \frac{\cos mx}{x^2+a^2} dx, a, m > 0. \tag{16}$$

To calculate the integral in Eq. (16), m is a constant, and the parameter t can still be introduced. That is, $I(t) = \int_{-\infty}^{+\infty} \frac{\cos tx}{x^2+a^2} dx$, $I(0) = \frac{\pi}{a}$. The derivative is found to be

$$I'(t) = -\int_{-\infty}^{+\infty} \frac{x \sin tx}{x^2+a^2} dx = -\int_{-\infty}^{+\infty} \frac{(x^2+a^2-a^2)\sin tx}{x(x^2+a^2)} dx = a^2 \int_{-\infty}^{+\infty} \frac{\sin tx}{x(x^2+4)} dx - \pi. \tag{17}$$

According to the following formula it does not get a specific value after taking the first derivative, but know that when t = 0, I'(0) = -π. In this case, next can take the second derivative

$$I''(t) = a^2 \int_{-\infty}^{+\infty} \frac{\cos tx}{x^2+a^2} dx = a^2 I(t) \tag{18}$$

The second order homogeneous ordinary differential equation can be obtained $I''(t) - a^2 I(t) = 0$. The general solution is $I(t) = C_1 e^{at} + C_2 e^{-at}$. Let $I(0) = \frac{\pi}{a}$, $I'(0) = -\pi$ into $I(t)$, one finds that

$$C_1 + C_2 = \frac{\pi}{a}, aC_1 - aC_2 = -\pi. \tag{19}$$

By solving for two undetermined constants $C_1 = 0$ and $C_2 = \frac{\pi}{a}$. Hence, it is solved that $I(t) = \frac{\pi}{a} e^{-at}$. Let $t = m$, $I = \frac{\pi}{a} e^{-mt}$. Thus, the value in Eq. (16) is $I = \frac{\pi}{a} e^{-mt}$.

3.5. Example 5

The fifth integral is

$$I(\xi) = \int_0^{+\infty} xe^{-x^2} \sin \xi x \, dx. \tag{20}$$

To begin with, the author will introduce the function $f(\xi) = -\int_0^{+\infty} e^{-x^2} \cos \xi x \, dx$. It can know that the relationship between the integral of the original function and the new function is

$$I(\xi) = f'(\xi) = \int_0^{+\infty} xe^{-x^2} \sin \xi x \, dx \tag{21}$$

It is known by the parity of the integrand that $\int_0^{+\infty} xe^{-x^2} \sin \xi x \, dx = 0$, hence $f'(\xi) = \frac{1}{2}\int_0^{+\infty} xe^{-x^2} \sin \xi x \, dx = \frac{1}{2}\int_0^{+\infty} xe^{-x^2} (\sin \xi x + \cos \xi x) \, dx$. Therefore,

$$f'(\xi) = \frac{\xi}{4}\int_{-\infty}^{+\infty} e^{-x^2} \cos \xi x \, dx = -\frac{\xi}{2} f(\xi) . \tag{22}$$

The general solution of this ordinary differential equation is $f(\xi) = Ce^{-\frac{\xi^2}{4}}$, and the integral constant is

$$f(0) = C = -\int_0^{+\infty} e^{-x^2} \, dx = -\frac{\sqrt{\pi}}{2} \tag{23}$$

Bring back the original function to get the analytic expression $f(\xi) = -\frac{\sqrt{\pi}}{2} e^{-\frac{\xi^2}{4}}$. Finally, the original integral is

$$I(\xi) = f'(\xi) = -\frac{\sqrt{\pi}}{4} \xi e^{-\frac{\xi^2}{4}} \tag{24}$$

Thus, the value in Eq. (20) is $I(\xi) = f'(\xi) = -\frac{\sqrt{\pi}}{4} \xi e^{-\frac{\xi^2}{4}}$.

3.6. Example 6

The sixth integral is

$$I = \int_0^{\infty} \frac{\ln cos^2 x}{x^2} dx. \tag{25}$$

To calculate the integral in Eq. (25), one notes that

$$2I = \int_{-\infty}^{\infty} \frac{\ln(cos^2 x)}{x^2} dx = \sum_{n=-\infty}^{+\infty} (\int_{n\pi}^{(n+1)\pi} \frac{\ln(cos^2 x)}{x^2} dx) = \sum_{n=-\infty}^{+\infty} (\int_0^{\pi} \frac{\ln(cos^2 x)}{(x+n\pi)^2} dx). \tag{26}$$

In other words, it is found that

$$I = \int_0^{\pi} (\sum_{n=-\infty}^{+\infty} \frac{1}{(x+n\pi)^2}) \ln (cos^2 x) \, dx = \int_0^{\pi} \frac{\ln(cos^2 x)}{sin^2 x} dx \tag{27}$$

where the interchange of the signs of integration and summation is justified by the fact that the integrands are all negative. Then, it is helpful to use the expansion $\sum_{n=-\infty}^{+\infty} \frac{1}{(x+n\pi)^2} = \frac{1}{sin^2 x}$. Using the symmetry of the integrand around the line $x=\pi/2x=\pi/2$, conclude that

$$I = \int_0^{\pi/2} \frac{\ln(cos^2 x)}{sin^2 x} dx = -\cot(x) \ln(cos^2 x) \Big|_0^{\pi/2} + \int_0^{\frac{\pi}{2}} \cot(x) \frac{-2 \cos x \sin x}{cos^2 x} dx = -\pi. \tag{28}$$

Thus, the value in Eq. (25) is $-\pi$.

4. Conclusion

In general, for Feynman integrals, the scope of application mainly depends on the nature of the integrable function and the complexity of the integration problem. For some complex integrals that cannot be solved by simple transformations or known formulas, Feynman integration may not be directly applicable or have limited effects. Feynman integral has many advantages, such as its high flexibility, which is especially suitable for some traditional integration problems that are difficult to deal with. In some cases, the calculation process can be simplified, and the integral result can be solved quickly by constructing the auxiliary function skillfully and using the known formula. Moreover, it has a wide range of applications, and it has applications in multiple fields. Feynman integrals also have a number of disadvantages, such as limited scope of application, as mentioned before, Feynman integral method can not be applied to all integrals. For some complex integral problems that cannot be solved by simple transformations, Feynman integral method may not be able to give effective solutions. Feynman integral method requires high mathematical literacy and problem-solving skills. It can be difficult for beginners to master and apply this method. Therefore, it is necessary to have a certain mathematical foundation and problem-solving experience in practical application. There is also a high risk of introducing errors in numerical calculations.

With the continuous development of mathematics and computing techniques, the methods of integral calculation are also constantly evolving and improving. In the future, people can expect more innovative integral calculation methods to emerge to deal with more complex and diverse integral problems. For Feynman integration, although its scope of application is limited and technical, it still has unique advantages and values in specific fields and specific problems. Therefore, in the future study and research, this method can continue to be deeply explored and applied to expand its application field and improve computing efficiency.

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