

# Uncertainty principle and complementary variables

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**Abstract.** The two most important areas in modern physics are quantum mechanics and the theory of relativity. Unlike classical physics, where Newton's mechanics dominates, these two areas change human beings' fundamental view of the universe. One of the theories that build up the base of quantum mechanics is Heisenberg's Uncertainty Principle. Starting from a thought experiment, Heisenberg's microscope in the setting of classical physics, Werner Heisenberg built a bridge between classical and quantum physics by presenting a counterintuitive outcome in the thought experiment. Since then, the observer of a physics phenomenon is no longer a bystander. The behavior of observation became a part of the physical experiment. To come up with a mathematical expression that can describe such a new discovery, Heisenberg came up with matrix mechanics and the concept of complementary variables. There is a trade-off between a pair of complementary variables. When one of them is measured precisely, meaning the information of that variable is known on a large scale, the other variable can not be measured precisely, meaning there is no way to know enough information about the other variable. The principle indicates a fundamental limit on what human beings can know about the unknown variables. The discoveries of other complementary variables help physicists know the new image of the physics world under new rules.

**Keywords:** Enter key words or phrases in alphabetical order, separated by commas.

## 1. Introduction

In the 1900s, phenomena that classical physics could not explain started to appear at a minuscule scale. To model those phenomena and set new rules that govern the microscopic world, physicists explored and established new fundamental rules. One of the examples that subvert the previous understanding of classical physics is Heisenberg's Uncertainty Principle. In classical physics, objects are treated as rigid bodies. The physical quantities of a rigid body in classical physics can easily be measured, such as momentum and position. However, in quantum physics, where particles are measured at the smallest scale possible, observation starts to affect measurement precision. Namely, the behavior of observation itself, which is essential to measurements, changes the physical quantities of the smallest particles and limits the number of physical quantities that can be precisely measured at one time. To elaborate, when light is being used to measure the momentum and position of a particle, the two quantities cannot be precisely measured at the same time. When the particle's position is precisely measured, its momentum is changed because photons of the light have momentum themselves and collide with the particle during measurement. Thus, the precise momentum of the particle cannot be precisely measured since it has altered. Vice versa, when the light of a longer wavelength is used to precisely measure the particle's momentum, its position cannot be precisely measured. Using a longer-wavelength light, the particle's position cannot be located in the range of the distance between two light wave crests because the wavelength of the matter wave, namely the De Broglie wave, of the particle is longer than the wavelength of light [1].

The physical quantities momentum and position stated above are called complementary variables. The math equation expresses this principle is as follows:

$$\Delta x \Delta p \geq \hbar/2 \quad (1)$$

As expressed in the equation, there is a minimal value for the product of momentum and position. This illustrates that if momentum gets larger, the value of the position will decrease. Vice versa, the value of momentum will decrease when the value of the position gets larger.

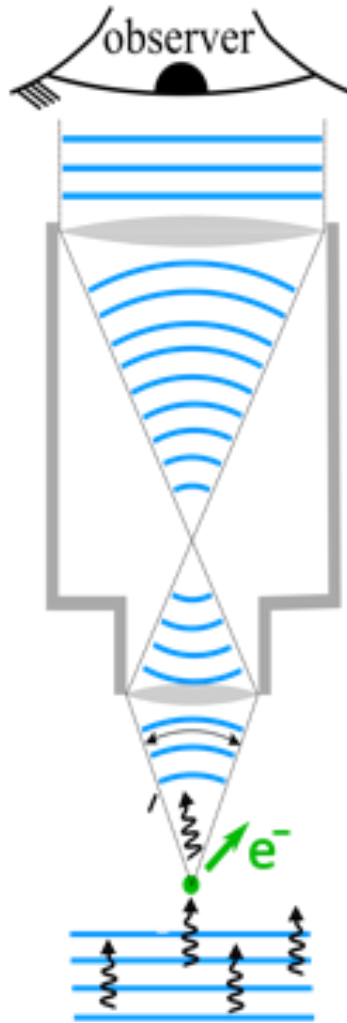
A pair of variables, such as momentum and position, are known as complementary variables. Basically, complementary variables cannot be precisely measured at the same time. Discussion on other complementary variables will be made later in the paper.

George Greenstein and Arthur Zajonc, in their book *The Quantum Challenge: Modern Research on the Foundations of Quantum Mechanics*, engaged extraordinary phenomena of quantum mechanics and enormous challenges people faced in the world of quantum mechanics. In this paper, the discussion of determining the electron's position based on the explanation of matter wave when momentum is precisely measured is cited. In his book *The Physical Principles of the Quantum Theory*, Werner Heisenberg illustrated the very first image of the quantum mechanics world around 1930. In this paper, his famous thought experiment Heisenberg's microscope in the book is cited to illustrate the initial intuition of the Uncertainty Principle. Also, Michael Richmond used to give a straightforward description of Heisenberg's Microscope. In detail, he expresses the quantum mechanics property under a seemingly classical physics thought experiment in the book *Theory of Linear Operators in Hilbert Space* by N.I. Akhiezer and I.M. Glazman carefully explained the details of operators in mathematics. Operators are important in the uncertainty principle because self-adjoint operators represent complementary variables. In short, the operator is an essential math concept to develop matrix mechanics. L.D. Landau and E.M. Lifshitz have a great explanation of the trade-off between complementary variables in their book *Quantum Mechanics: Non-Relativistic Theory*. The concept of eigenstate is mentioned during the process, which helps the readers intuitively visualize the idea behind the uncertainty principle. Steve Mann and Simon Haykin explained in detail in their research paper, the Chirplet Transform: Physical Considerations, how canonically complementary variables, time and frequency form complementary variables. The behavior of measurement on one of them will affect the one on the other. Tianyuan Cao gave much interesting intuition in his book *History of quantum physics*, which are elaborated and extended in this paper. Basically, it is about the deeper indication of the discovery of complementary variables. Michael Nolle and Martin Suda stated an application of complementary variables in their research paper. The underlying mathematical mechanism makes signal and image processing advance. Miroslav Svítek, in his paper *Complementary variables and its application in statistics*, discussed the application of complementary variables in the field of statistics. Rather than the previous example of signal and image processing, which mainly applies the mechanism of the mathematical tools used, the example in statistics indicates a more straightforward use of the properties of complementary variables itself. Jakub Czajko mentioned in his research paper that advance in measurement tool helps us refine mathematical expressions to establish more accurate theories. It is worth being cited for having a peek at the foresight of complementary variables.

## **2. Mechanism**

### **2.1. Heisenberg's microscope**

All physics equations start with intuition to describe a phenomenon. For the Uncertainty Principle, Heisenberg started with a thought experiment known as Heisenberg's microscope. To study the effect of observation on measurement, Heisenberg treated electrons as particles.



**Figure 1.** Heisenberg's microscope thought experiment.

Imagine a microscope is set up to observe an electron, and the electron is put at the lens's focus through which light rays exits and bounce back. Thus, it is ready to observe the position of the electron. Let the wavelength of the light rays be  $\lambda$ , and the angle between the light rays and the electron be  $\theta$ . According to optics in classical physics, the position of the electron cannot be measured precisely. There will be a margin of error equals to [2, 3]

$$\Delta x = \lambda / \sin \theta. \quad (2)$$

On the other hand, different from observing a rigid body in the general world, at a minuscule level, photons of the light rays themselves will collide with the electron. Based on Compton recoil, the electron will gain a momentum proportional to  $h/\lambda$  (Planck's constant over the wavelength of the light rays). Nevertheless, because countless light rays are shot from the light resource, the direction of the resulting momentum gained by the electron can only be measured to a precision of [3]

$$\Delta p \approx h \sin \theta / \lambda. \quad (3)$$

Putting (2) and (3) together, a general equation of Heisenberg's uncertainty principle is expressed, which is quite similar to the final exact equation (1).

## 2.2. Matrix mechanics

In classical mechanics, Newton's first, second, and third laws are used to model the behavior of a rigid body in the system. However, having more complex properties, quantum mechanics needs a new set of rules to illustrate the quantum world properly. Werner Heisenberg, Max Born and Pascual

Jordan in 1925 came up with matrix mechanics to model the phenomena that happened at the quantum scale.

### 2.2.1. self-adjoint operators.

It is known that position  $x$  and momentum  $p$  in section 2.1 are complementary variables. In matrix mechanics, they are expressed as self-adjoint operators. A self-adjoint operator on an infinite-dimensional complex vector space  $V$  with inner product  $\langle x, y \rangle$  is a linear map  $A$  that is its own adjoint [4].

### 2.2.2. Commutator After a pair of complimentary variables $\hat{A}$ and $\hat{B}$ are defined as self-adjoint operators, their commutator is defined as

$$[\hat{A}, \hat{B}] = \hat{A} \hat{B} - \hat{B} \hat{A}. \quad (4)$$

Take position  $x$  and momentum  $p$  in section 2.1 as an example again, after being treated as self-adjoint operators, the commutator of them is the canonical commutation relation

$$[\hat{x}, \hat{p}] = i\hbar. \quad (5)$$

The effect of the commutator on position and momentum eigenstates can illustrate the meaning of non-commutativity. Suppose  $|\psi\rangle$  is a right eigenstate of position with constant eigenvalue  $x_0$ . The effect of commutator to  $|\psi\rangle$  will result in

$$[\hat{x}, \hat{p}] |\psi\rangle = (\hat{x}\hat{p} - \hat{p}\hat{x}) |\psi\rangle = (\hat{x} - x_0 \hat{I}) \hat{p} |\psi\rangle = i\hbar |\psi\rangle. \quad (6)$$

Let  $|\psi\rangle$  also be a right eigenstate of momentum with constant eigenvalue  $p_0$ , the result below can be gained:

$$(\hat{x} - x_0 \hat{I}) \hat{p} |\psi\rangle = (\hat{x} - x_0 \hat{I}) p_0 |\psi\rangle = (x_0 \hat{I} - x_0 \hat{I}) p_0 |\psi\rangle = 0. \quad (7)$$

This commutation relation needs to have the condition of

$$[\hat{x}, \hat{p}] |\psi\rangle = i\hbar |\psi\rangle \neq 0, \quad (8)$$

Which means that no quantum state can simultaneously be both a position and a momentum eigenstate [5].

When a state is measured, it is projected onto an eigenstate based on the relevant observable. For example, if a particle's position is measured, then the state amounts to a position eigenstate. This means that the state is not a momentum eigenstate, however, but rather it can be represented as a sum of multiple momentum basis eigenstates. In other words, the momentum must be less precise. As in the wave mechanics interpretation above, one sees a tradeoff between the respective precision of the two, quantified by the uncertainty principle [5].

### 2.3. Other complementary variables

As a pair of variables position  $x$  and momentum  $p$  in section 2.1, they are called complementary variables. When one of two complementary variables is measured precisely, the other one cannot be measured precisely. In fact, other than position  $x$  and momentum  $p$  discussed above, there are many other pairs of complementary variables. Other complementary variables include canonically complementary variables and a pair of variables that are the derivative of the other. An example of canonically complementary variables is time and frequency (if the frequency of a long musical note is measured precisely, the precision on the measurement of time duration is reduced. On the other hand, the measurement of the duration of a short musical note is very precise, but the precision on the measurement of its frequency diminishes) [6]. For complimentary variables of derivatives of action, examples are the energy of a particle at a certain event and the time of the event, angular momentum and angular position (similar to momentum and position), the Newtonian gravitational potential at an event and the mass density at that event, the electric/magnetic potential at an event and the density of electric charge/current at that event.

### 3. Result

In the mechanism part, we have gone through the entire process of the birth of Heisenberg's uncertainty principle. One of the most important indications is that there is a certain limit on the precision of measurement we can get for a pair of complementary variables. However, can we get any other indication from complementary variables? Are there any applications of the complementary variables? Is it possible to discover more complementary variables? Despite the phenomenon that there is a trade-off between complementary variables, we seek a more physical explanation. In the case of momentum  $p$  and position  $x$ , the physical interpretation is that we cannot know every single aspect of the electron. Also, this means that when the electron's momentum is measured precisely, the electron will appear at a very extreme position (By extreme, it means a very distant location). Vice versa, the electron will have a wide range of momentum values when its position is measured precisely. In the same way, physical interpretation of other complementary variables also gives us a peek at the real physical world that we cannot see with our eyes. The other example is the energy of a particle at a certain event and the time duration of the event. On the surface, the straightforward interpretation is that precise measurement of the time duration of the event will cause a large uncertainty on the measurement of the particle's energy. However, the deep physical meaning behind that is interesting. Based on the superficial interpretation, the energy can fluctuate greatly in a very small amount of time [7]. Although we cannot see the phenomenon directly, there is a possibility that the energy of an event jumps to an extremely high value in 1/100 second. Take an example of canonically conjugate variables, the physical interpretation of the uncertainty on doppler and range is also fascinating. The superficial interpretation is that the more we know about how far away a radar target is, the less we can know about the exact velocity of approach. Quite similar to momentum and position, when we know exactly the location of the radar target, the physical interpretation of the uncertainty in exact velocity will enable the radar target to have a probability of having an extremely fast speed. Although we cannot visually sense that, the uncertainty principle indicates an unbelievable picture of the world we can see. It sheds light on Albert Einstein's famous quote, "Is the moon there when no one looks?". Some may argue that it must be there for distant planets, such as Neptune, although it is far away from us. However, note that "see" here does not necessarily mean visually sense. As long as a certain type of observation is made, we categorize it as "see". In the Neptune case, it constantly affects the earth's gravitational field, so its existence is ensured.

Complementary variables also have applications in many fields. To elaborate, in the field of signal and image processing, the transition probability of quantum states leads to a distance function on the signals [8]. The mathematical property of complementary variables helps different lines of information processing work simultaneously. Also, in the field of statistics, pair of complementary variables has its own use. For applied probabilistic modeling, by introducing the complementarity's principle between  $x$ -representation (random time series, random process) and  $p$ -representation or  $k$ -representation (rate of change/velocity of random time series and processes) and probability theory is completed for the "structural" parameter, which carries information about the changes of studied time series or the random process [9].

A more profound question is whether we could find more complementary variables. In the era of classical physics, because the tools for measurement were way less precise than today's, many theories were simply accepted. Take Newton's gravitational equation as an example, and it seemed universal across the universe until Einstein introduced his famous relativity theory. The refinement of the measurement tools enables people to spot the flaws of Newton's gravitational theory and finally proves that Einstein was right. Also, advance in technology allows scientists to detect gravitational waves predicted by Albert Einstein. As stated in the mechanism part, the difference in experimental and predicted values by math equations indicates small problems in current theories. With the refinement and advances in measurement tools, more complementary variables can be discovered. Thus, we can refine mathematical expressions to come up with a better theory [10].

## 4. Conclusion

The uncertainty Principle signals a fundamental shift from classical physics to modern physics, namely Newton's mechanics to Quantum mechanics. In the world of quantum mechanics, the position of a particle cannot be precisely determined according to the Uncertainty Principle. Instead, the position of a particle at the quantum level is described using a wave function, illustrating the possibility of the particle appearing at one location. As Albert Einstein's elevator thought experiment, which led to the establishment of special relativity, the uncertainty principle starts with Heisenberg's famous thought experiment Heisenberg's microscope. Commonly, we find new theories unreasonable because all of our consciousness are built on previous theories. However, the trade-off of our struggling mind is a brand-new field of applications and a step closer to the truth.

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