Research on Integrated application of Advance Mathematics and linear Algebra

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Abstract. Advanced Mathematics and Linear Algebra are the two most basic and important public subjects of science and engineering in universities. In general, the main content of Advanced mathematics is calculus, which belongs to the type of analysis (analysis and study of the local and global properties of continuous objects). Linear algebra, on the other hand, is a course characterized by algebra (algebra studies the structural properties of discrete objects). From the nature of the two courses, they have great differences in thinking mode and processing methods. Students often to think that Advanced mathematics and Linear algebra are two relatively independent systems. However, after systematically study Advanced mathematics and Linear algebra, I found that Advanced mathematics (mainly calculus) and Linear algebra, two relatively independent courses, also have many methods and ideas that can be borrowed from each other. I shall discuss the relationship between Advanced mathematics (calculus) and Linear algebra in three ways.

Keywords: Advanced Mathematics, Linear Algebra, Fibonacci sequence, Proof of Lagrange's mean value theorem, Determinant, Auxiliary matrix, interpenetrating.

1. Main body

A: Two courses can solve the different problem with same ideas

It's going to show two examples:

Question1: prove that any function defined on a symmetric interval about the origin can always be represented as the sum of an odd and an even function.

We make $f(x)$ is defined on the interval L that is symmetric about the origin.

Then, we can get that:

$$g(x) = \frac{1}{2}[f(x) + f(-x)]$$

is the even function on L.

$$h(x) = \frac{1}{2}[f(x) - f(-x)]$$

is the odd function on L.

Obviously

$$f(x) = g(x) + h(x)$$

Question2: proved that any real matrix of order n can be expressed as the sum of a symmetric matrix and an antisymmetric matrix.

We make that A is a real matrix of order n.

and then:

$$\boldsymbol{M} = \frac{1}{2}(A + A^T)$$

is a symmetric matrix.

$$\boldsymbol{N} = \frac{1}{2}(A - A^T)$$

is an antisymmetric matrix.

Obviously

$$A = M + N$$

As we can see from these two examples, the first one is a problem of Advanced mathematics, and the second one is a problem of Linear algebra. Although the two questions are different in form, through the analysis of the above solution methods, we can clearly see that their solution ideas are similar (both questions use the idea of construction)

B: Two courses can solve the same problem with different ideas

Question: The sequence $\boldsymbol{F_1, F_2, \cdots, F_n, \cdots}$ which satisfies that $\boldsymbol{F_1 = F_2 = 1, F_n = F_{n-1} + F_{n-2}, (n \geq 3)}$. Making that $\boldsymbol{a_n = F_{n+1} \div F_n}$, prove that there is the limit for $\boldsymbol{\lim_{n\to\infty} a_n}$,and what is the limit for it.

Solution 1(From the point of view of Advanced mathematics):

Because

$$\boldsymbol{a_n = \frac{F_{n+1}}{F_n} = \frac{F_{n-1} + F_n}{F_n} = 1 + \frac{F_{n-1}}{F_n} = 1 + \frac{1}{a_{n-1}}}$$

and by analysis we can find that the whole sequence is not monotonic, but if we separate the odd and even terms, each new sequence seems to be monotonic. Now let's prove it by mathematical induction:

proving of monotonic:

Because $a_1 < a_3, a_2 > a_4$. we make that $a_{2n-1} < a_{2n+1}$. Then we can have that

$$a_{2n-1} = 1 + \frac{1}{a_{2n-1}} \geq 1 + \frac{1}{a_{2n+1}} = a_{2n+2}$$

$$a_{2n+1} = 1 + \frac{1}{a_{2n}} \leq 1 + \frac{1}{a_{2n+2}} = a_{2n+3}$$

So, we can get that $\{a_{2n-1}\}$ is monotonically increasing and $\{a_{2n}\}$ is monotonically decreasing.

proving of boundedness:

Because $a_1 = 1, a_2 = 2$, so $1 \leq a_1, a_2 \leq 2$. We make that $1 \leq a_n \leq 2$, then $1 \leq a_{n+1} \leq 2$ (because $a_{n+1} = 1 + \frac{1}{a_n}$). By using the mathematical induction, we can get that $\{a_{2n-1}\}, \{a_{2n}\}$ is monotonic and has boundedness, which means that it has the limit. We make that:

$$\lim_{n\to\infty} a_{2n-1} = A, \lim_{n\to\infty} a_{2n-1} = B,$$

according to the recursion formula of $\{a_n\}$ and takes the limit of both sides we can get that:

$$A = 1 + \frac{1}{B}, B = 1 + \frac{1}{A}$$

$$So \ \ A = B = \frac{1 + \sqrt{5}}{2}$$

$$Means\ that\ \lim_{n\to\infty} a_n = \frac{1 + \sqrt{5}}{2}$$

Solution 2(From a Linear algebra point of view):

From the perspective of Linear algebra, we use the idea of subspace to solve:

We make that:

$$\beta = (b_1, b_2, b_3, \cdots, b_n) \in R^n, (n \geq 3),$$

then we make the set which consisting of a whole sequence of numbers that satisfies $b_n = b_{n-1} + b_{n-2}$ to be $W \subset R^n$, and it is easy for us to test this kind of sequence is closed to addition and multiplication. So, it is the subspace of R^n.

We can know that the first two items b_1, b_2 which belong to $\beta = (b_1, b_2, b_3, \cdots, b_n) \in W$ could take any real value we want, and the following items $b_n (n \geq 3)$ is completely determined by the first two terms. So, (b_1, b_2) could be considered as the coordinate of $(b_1, b_2, b_3, \cdots, b_n) \in W$, where W is a two- dimensional subspace. We choose $\beta_i = (1, q_i, q_i^2 \cdots, q_i^{n-1})(i = 1,2)$ to be a basis of W. Then, $Fibonacci$ sequence $\varphi = (F_1, F_2, \cdots, F_n) \in W$ could be decomposed into a linear combination of this basis $\varphi = x\beta_1 + y\beta_2$. We know that the general formula of geometric series is $b_n = b_1 q^{n-1}$, then the necessary and sufficient for geometric sequence $\beta \in W$ is $b_n = b_{n-1} + b_{n-2}$ which means that:

$$b_1 q^{n-1} = b_1 q^{n-2} + b_1 q^{n-3} (n \geq 3)$$

and the two roots of $q^2 = q + 1$ are:

$$q_1 = \frac{1 - \sqrt{5}}{2}, q_2 = \frac{1 + \sqrt{5}}{2}$$

It is obviously that $\beta_1 = (1, q_1, q_1^2 \cdots, q_1^{n-1})$ and $\beta_2 = (1, q_2, q_2^2 \cdots, q_2^{n-1})$ are linearly independent, and it could be a basis of W:

$$(F_1, F_2, \cdots) = x(1, q_1 \cdots) + y(1, q_2 \cdots)$$

$$= (x + y, xq_1 + yq_2, \cdots)$$

From the initial condition we can get that:

$$F_n = \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}}$$

So

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{F_{n-1}}{F_n} = \frac{1 + \sqrt{5}}{2}$$

From these two examples, we can get that we can solve the general term formula of $Fibonacci$ sequence from the perspective of Advanced mathematics, and it is also a good solution to start from the perspective of Linear algebra. From this we can see that Advanced mathematics and Linear algebra can give different solutions to the same problem, the two are interpenetrating.

C: Combining the two Courses to solve the problem

$Lagrange's\ mean\ value\ theorem$ is a particularly important theorem in mathematics, and we usually prove $Lagrange's\ mean\ value\ theorem$ in terms of $Rolle's\ theorem$ by constructing auxiliary functions:

$$\varphi(x) = f(x) - f(a) + \frac{f(b) + f(a)}{b - a}(x - a)$$

This is the way to prove it through the ideas of Advanced mathematics.

Then, the next we will prove $Lagrange's\ mean\ value\ theorem$ by using the method that combining the ideas of Advanced mathematics and Linear algebra.

Firstly, we get the $Lagrange's\ mean\ value\ theorem$:

Make the function which satisfies that 1: Continuous on the closed interval $[a, b]$, 2: differentiates on the open interval (a, b), then there must be at least one point $\xi \in (a, b)$ could make that $f'(\xi) = \frac{f(b)-f(a)}{b-a}$.

Then, we propose an auxiliary theorem:

To get the determinant $\begin{vmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t) \end{vmatrix}$, where $a_{ij}(t)$ is continuously differentiable and $i,j = 1,2,3$.

Then we compute it by expanding the determinant, we can get that:

$$\frac{d}{dt}\begin{vmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t) \end{vmatrix}$$

$$= \begin{vmatrix} a'_{11}(t) & a_{12}(t) & a_{13}(t) \\ a'_{21}(t) & a_{22}(t) & a_{23}(t) \\ a'_{31}(t) & a_{32}(t) & a_{33}(t) \end{vmatrix} + \begin{vmatrix} a_{11}(t) & a'_{12}(t) & a_{13}(t) \\ a_{21}(t) & a'_{22}(t) & a_{23}(t) \\ a_{31}(t) & a'_{32}(t) & a_{33}(t) \end{vmatrix} + \begin{vmatrix} a_{11}(t) & a_{12}(t) & a'_{13}(t) \\ a_{21}(t) & a_{22}(t) & a'_{23}(t) \\ a_{31}(t) & a_{32}(t) & a'_{33}(t) \end{vmatrix}$$

Now we prove $Lagrange's\ mean\ value\ theorem$:

We make the auxiliary functions $F(x) = \begin{vmatrix} 1 & 1 & 1 \\ x & a & b \\ f(x) & f(a) & f(b) \end{vmatrix}$, by using the auxiliary theorem we can get that:

$$F'(x) = \frac{dF(x)}{dx}\begin{vmatrix} 1 & 1 & 1 \\ x & a & b \\ f(x) & f(a) & f(b) \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & 1 \\ 1 & a & b \\ f'(x) & f(a) & f(b) \end{vmatrix} + \begin{vmatrix} 1 & 0 & 1 \\ x & 0 & b \\ f(x) & 0 & f(b) \end{vmatrix} + \begin{vmatrix} 1 & 1 & 0 \\ x & a & 0 \\ f(x) & f(a) & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & 1 \\ 1 & a & b \\ f'(x) & f(a) & f(b) \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ 1 & a & b-a \\ f'(x) & f(a) & f(b)-f(a) \end{vmatrix}$$

$$= f'(x)(b-a) - \big(f(b) - f(a)\big)$$

Means

$$f'(\xi)(b-a) - \big(f(b) - f(a)\big) = 0$$

Then, $F(x)$ continuous on the closed interval $[a,b]$ and differentiates on the open interval (a,b), by the time, $F(a) = F(b) = 0$. So, for $F(x)$ there must be one point $\xi \in (a,b)$ could make that $F'(\xi)$ =0.

From this proof, we can see that the knowledge of Advanced mathematics and Linear algebra can be integrated with each other. Sometimes, unexpected results may be achieved by using the idea of integration of the two courses to solve problems.

2. Conclusion

Viewed from the above three aspects, Advanced mathematics and Linear algebra, which are the courses make people feel not very relevant, actually have a lot of integration and can learn from each other. We should continue to learn more professional knowledge to expand our knowledge base. At the same time, we should also think more about whether there is some continuity between different courses. If we make relevant connections between different courses and understand them well, we can often get twice the result with half the effort.

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