Theory and Application of Compton Scattering Experiment

Jiang Zhu*

Ulink College of Shanghai, Shanghai, China

*Corresponding author: jiang.zhu@ulink.cn

Abstract. Compton scattering has been a key concept in atomic and molecular physics, material science, condensed matter physics, and other fields ever since it was originally discovered by Arthur H. Compton in 1923. Additionally, the Compton camera, one of the applications of Compton scattering can gather sufficient data and information about photons with energies above 500 keV, which is important for scientific research into astronomy, medical imaging, and the visualization of radioactive materials. The free electron approximation, the impulse approximation, and the scattering matrix are some of the methods used to arrive at the Compton formula and the underlying principles of the Compton effect. In this article, a full derivation of Compton formula will be included, along with a deduction of the free electron approximation, which shows the relationship between Compton scattering and Thomson scattering, a low-energy limit of the former when the photon energy is much less than the mass energy of the particle. Also, the article will discuss several thoughts of Compton scattering, including the examination of the connection between wavelengths and relative intensities, the defiance of conservation laws, and virtual photon absorption.

Keywords: Compton scattering; Free electron approximation; Planck-Einstein relation; Quantum theory.

1. Introduction

Proposed by J. J. Thomson in 1904, the classical Thomson scattering [1] describes the scattering of light with low intensity by electrons as a linear process. This theory, mainly based on the electrodynamics, leads to the conclusion that neither the radiation's frequency is changed, nor is the light's magnetic-field component involved in the scattering process [2]. However, though it succeeds in explaining the results from experiments regarding to X-rays of moderate hardness, several defects appear when X-rays of greater hardness or even γ-rays are used, in which case, the experimental value is far less than the prediction from Thomson scattering theory [3]. Later in the century in 1923, Arthur H. Compton put forward the theory of Compton effect, also known as the Compton scattering. Compton scattering is the scattering of a bound electron in atomic orbital by an incidence of a high-frequency photon (generally an X-ray), which results in a recoil of the electron and a decrease in photon's wavelength. This phenomenon can be expressed as $e + \gamma \to e + \gamma'$, where e represents the bound electron, γ the incident photon and γ' the scattered photon [4]. The quantitative relationship between the wavelengths of incident and scattered photon can be concluded by using the Compton formula, which will be elaborated in detail later.

Since the very first time when Compton scattering was discovered and Compton formula was proposed, there have been other invaluable approaches that create theoretical techniques to the theory by using the ab initio calculations [4]. These methods utilize the intrinsic link of Compton effect with electronic momentum density [5]. The Free Electron Approximation (FEA), created and developed in 1929 by O. Klein and Y. Nishina, is one of them. This approach views the so-called bound electron as completely free, free of any atomic binding, shielding effect, or many-body interactions. [6]. Another approach is the Impulse Approximation, which can be further categorized into non-relativistic impulse approximation by DuMond [7] and relativistic impulse approximation by R. Ribberfors et al. [8]. In recent years, Compton scattering has been applied in a wide range of subjects and sciences, including astrophysics, condensed matter physics and nuclear physics etc. It is also used in other fields such as medical imaging [9] and tomography system [10].

2. Derivation of Compton Formula

Firstly, the article will first focus on several relations, that can be of great use in terms of analyzing the Compton scattering quantitatively. The first one is the energy-momentum relation, proposed by Einstein, is given as follows [11]:

$$E^2 = (pc)^2 + (mc^2)^2 \tag{1}$$

Where E represents the total energy, p the momentum and m the mass of the object. The second relation is the Planck-Einstein relation, which is also known as the energy-frequency relation. According to the wave-particle duality for electromagnetic waves, the incident X-ray can be considered to propagate as both waves and photons [12]. In this instance, the photon has energy E and frequency υ. The two quantities can be related by using the formula

$$E = h\upsilon, \tag{2}$$

Where h is the Planck's constant. Given $c = \lambda\upsilon$, Eqn. (2) can be re-formulated into

$$E = \frac{hc}{\lambda}. \tag{3}$$

In a classical point of view, the scattering process can be considered as an incidence of a particle on some scattering matter [13]. One key concept in scattering is known as the differential cross-section, defined by $D(\theta) \equiv \frac{d\sigma}{d\Omega}$. In this case, when the incident particle is incident within an infinitesimal cross-sectional area $d\sigma$, it will scatter in a range of infinitesimal solid angle $d\Omega$, which is proportional to $d\sigma$ with the proportionality factor $D(\theta)$.

By contrast, in the quantum scattering theory, the main issue is to solve for the scattering amplitude $f(\theta)$, which gives the probability of scattering in one specific direction θ [14]. Similarly, the differential cross-section $D(\theta)$ is also involved, relating to the scattering amplitude by the formula $D(\theta) \equiv \frac{d\sigma}{d\Omega} = |f(\theta)|^2$. One important figure of this equation is the relationship between the differential cross-section and scattering amplitude, with the former being experimentally acquired and the latter derived from Schrödinger equation [14].

On top of the above, the derivation of Compton Formula can be achieved as below. The incident X-ray photon has the energy E_0 and momentum p_0. It hits the electron and thus scatters with angle of θ above the x-axis, with energy E_1 and momentum p_1. As a result of this scattering, the electron now moves in the direction of angle ϕ below the x-axis, with kinetic energy KE, total energy E and momentum p. The setup is shown in Fig. 1.

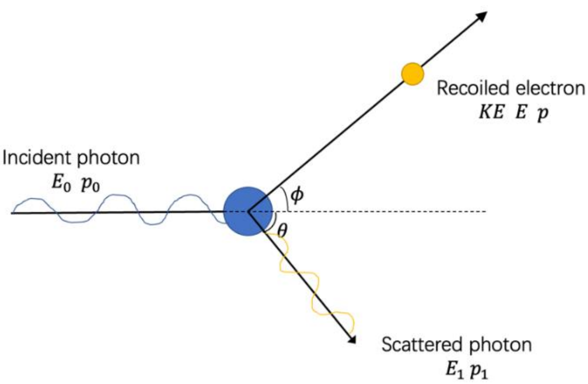

Fig. 1 Illustration of Compton Scattering

For one thing, due to the conservation of momentum, the initial momentum before collision is solely the momentum of incident photon p_0, and the final momentum is the sum of momenta of scattered photon and electron, $p_1 + p$. Thus, it gives

$$\boldsymbol{p_0} = \boldsymbol{p_1} + \boldsymbol{p} \tag{4}$$

Then, dividing the momenta into their x- and y-components respectively yields $p_0 = p_1 \cos\theta + p\cos\phi$ in x-direction, and $p_1 \sin\theta - p\sin\phi = 0$ in y-direction. Rearrange the two equations to separate momentum p's vertical and horizontal components: $p_0 - p_1\cos\theta = p\cos\phi$ and $p_1\sin\theta = p\sin\phi$. After that, squaring both sides of the equations above yields $p_0^2 - 2p_0p_1\cos\theta + (p_1\cos\theta)^2 = (p\cos\phi)^2$ and $(p_1\sin\theta)^2 = (p\sin\phi)^2$, and therefore $p_0^2 + p_1^2 - 2p_0p_1\cos\theta = p^2$. Applying the conservation of energy, the energy of the incident photon can be linked with that of scattered photon and electron, which can be formulated:

$$E_0 + m_e c^2 = E_1 + KE + m_e c^2 \tag{5}$$

Where the total energy $E = KE + m_e c^2$. Rearrange Eqn. (5), $E_0 - E_1 = E - m_e c^2$. Multiple both sides by c^2, and apply Eqn. (1) to substitute p^2c^2 with $E^2 - (mc^2)^2$; it is inferred that $E_0^2 + E_1^2 - 2E_0E_1\cos\theta = p^2c^2$ or $E_0^2 + E_1^2 - 2E_0E_1\cos\theta = E^2 - (mc^2)^2$. Taken together, it is found that

$$E_0E_1(1-\cos\theta) = (E_0 - E_1)m_e c^2. \tag{6}$$

In addition, using Planck-Einstein relation shown in Eqn. (3), the article can find the relation between energy and wavelength of both incident and scattered photons, $E_0 = \frac{hc}{\lambda_0}$ and $E_1 = \frac{hc}{\lambda_1}$. Substitution results in the Compton Formula

$$\lambda_1 - \lambda_0 \equiv \Delta\lambda = \frac{h}{m_e c}(1-\cos\theta) \tag{7}$$

Here, the term $\frac{h}{m_e c}$ is known as the Compton wavelength. For an electron, whose mass $m_e = 9.11 \times 10^{-31} kg$, the Compton wavelength is $\lambda_c = 0.0243 \times 10^{-10} m$

3. Free Electron Approximation

In FEA, all the many-body interactions, electron shielding effect and atomic binding effects in the system are ignored during the Compton scattering, and the so-called bound electron is considered to be stationary at a fixed position before the scattering process for simplicity [4]. O. Klein and Y. Nishina first introduced this FEA method in 1929 [6], which fully utilizes the Compton Formula in the energy form

$$\omega_f = \omega_c = \frac{\omega_i}{1 + \frac{\omega_i}{m_e c^2}(1-\cos\theta)} \tag{8}$$

Where ω_c is widely known as the Compton energy and θ is the scattering angle by which the scattered photon deviates from the line of incidence.

Due to the property of cosine function, the minimum value of ω_f can be found when $\theta = 180^\circ$; meanwhile the energy transfer, defined as $T = \omega_i - \omega_c$, reaches its maximum. This qualitative analysis matches the Compton edge in the $d\sigma/dT$ spectrum of Compton scattering [4]. The corresponding formulae are shown below

$$\omega_{c\,min} = \omega_{f\,min} = \frac{\omega_i}{1 + \frac{2\omega_i}{m_e c^2}} \tag{9}$$

And

$$T_{max} = \omega_i - \omega_{c\,min} = \omega_i - \frac{\omega_i}{1 + \frac{2\omega_i}{m_e c^2}} = \frac{2\omega_i^2}{m_e c^2 + 2\omega_i}. \tag{10}$$

O. Klein and Y. Nishina proposed the Klein-Nishina formula, which gives the differential cross-section of Compton scattering process in FEA [6,14]. The formula is expressed as

$$\left(\frac{d\sigma}{d\Omega_f}\right)_{FEA} = \frac{r_0^2}{2}\left(\frac{\omega_c}{\omega_i}\right)^2\left(\frac{\omega_i}{\omega_c} + \frac{\omega_c}{\omega_i} - (\sin\theta)^2\right) \tag{11}$$

In this equation, $r_0 = 2.8 \times 10^{-15}m$ is the classical radius of electron. It is noticeable that the cross section described by Eqn. (11) shows a significant characteristic of the spatial distribution of the scattered photon, that is being rotational symmetric with the line of incidence [5].

When the incident photon has photon-energy ω_i far less than mass-energy $m_e c^2$ in the non-relativistic limit, Eqn. (11) can be reduced to another form:

$$\left(\frac{d\sigma}{d\Omega_f}\right)_{Thomson} = \frac{r_0^2}{2}(1 + \cos^2\theta) \tag{12}$$

Furthermore, if the scattering occurs in the elastic limit when $\omega_f = \omega_c \to \omega_i$, the Compton scattering is reduced to Thomson scattering [4,5], as illustrated in Eqn. (12).

The article manages to plot the graphs of Eqn. (11) and (12), with the horizontal axis being the scattered angle θ and vertical axis being $\frac{d\sigma}{d\Omega_f}$. Compared to that of Eqn. (11), the graph of Eqn. (12) can be approximated to be a straight line close to 0, due to the coefficient $\left(\frac{\omega_c}{\omega_i}\right)^2$ in Eqn. (11). However, in fact, both graphs show a sinusoidal pattern due to the periodicity of the cosine function, with the minimum when θ is $90°$ and two maxima when θ is $0°$ and $90°$. This approximation reveals the nature of Thomson scattering, as it is only a low-energy limit of Compton scattering.

4. Analysis and Application

4.1. Graphical analysis of Compton scattering

To understand the effectiveness of Compton scattering with respect to the scattering angle, the article tries to plot a graph of $\Delta\lambda$, the change in wavelength, on the vertical axis against θ, the angle by which the scattered photon deviates from the line of incidence shown in Fig. 1. The graph can identify a symmetrical sinusoidal figure with a peak at $\theta = 90^\circ$ and zeros at $\theta = 0^\circ$ and $\theta = 180^\circ$.

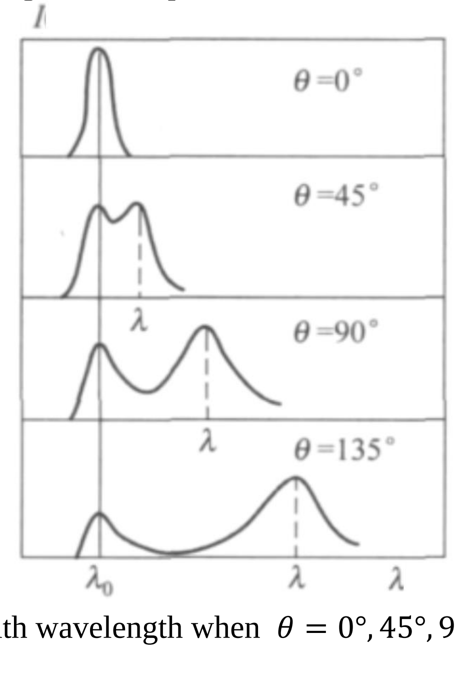

Fig. 2 Relative intensity with wavelength when $\theta = 0°, 45°, 90°, 135°$. From Ref. [15].

In light of the above, the article then investigates into the relationship between the intensity of the scattered X-ray, I, and the angle, θ. As considered in the FEA model, the electron is at rest without any binding or shielding effect. Thus, the intensity of the scattered EM wave can then be considered in terms of the classical theory [3]. Plotting the graph of relative intensity against wavelength shown in Fig. 2, a similar pattern can be revealed. The four diagrams all have a peak at incident wavelength λ_0, indicating that there is an unshifted beam shooting straightly with no deviation. This can be explained by the fact that the incident photon fails to eject an electron [15]. Thus, the size of the shift is determined by the Compton wavelength of the whole atom, which can be up to 10000 times smaller than the Compton wavelength of the electron. In addition to the peak at λ_0, an additional one can be seen at the scattered wavelength λ, when θ is other than 0. This peak can be attributed to the scattered beam whose wavelength can be given by Eqn. (7).

4.2. Contradiction to conservation laws and virtual photon absorption

The problem the article has been discussing so far is to deal with the phenomenon of increase in wavelength during the Compton scattering. It can be modelled as an elastic collision between two small balls. In Sec. 2, the article used the conservation of momenta and energy, shown as Eqn. to give the relations between the energy and momenta in x- and y-direction, and together with the relativistic energy formula, the Compton formula is derived in the form of Eqn. (7). Here comes the issue that since the change in wavelength $\Delta\lambda$ is a function of the scattering angle θ, which varies continuously in the given domain, it suggests that $\Delta\lambda$ can also vary continuously. This further allows the energy to change in a continuous pattern, which violates the quantum theory of light proposed by Einstein [16]. Hence, at least one of the two conservation rules must be abandoned when examining the elastic collision between a photon and an electron.

In order to solve this problem, the article applies an analogy to the formation of virtual phonon in electron-electron collisions in the field of solid physics [15]. Similarly, the photon-electron collision in the Compton scattering can also be viewed as a absorption and emission of virtual photon. Because of the reaction between photon and free electron in a short period of time, the energy of released photon differs from that of the incident one given by Eqn. (8), leading to the Compton effect.

5. Conclusion

As it has been such a significant finding, the Compton effect might be considered to be one of the cornerstones of quantum physics. Hitherto, the paper has demonstrated the derivation of Compton formula by using conservations of energy and momenta in the context of Free Electron Approximation. In this case, the model is simplified from the actual process in a way such that the electron is considered to be completely free and unbound, without any atomic binding, electron-electron repulsion effect, in the atomic orbital system. In addition, by using the free electron approximation method, the paper also prove that Thomson effect is the low-energy limit of the Compton effect, when the scatted photon energy is approximately equal to the incident photon energy. This also suggests that the Thomson scattering is an elastic collision without any energy loss, whereas the Compton scattering is an inelastic collision with the energy change. Moreover, the article investigated the relationship between the relative intensity and wavelength, whose graphs show two peaks each at original and shifted wavelengths shown in Fig. 2. Finally, the article suggests the existence of virtual photons, which play a vital role in explaining the contradiction between the continuous change in energy in Compton scattering and the quantum theory of light by Einstein. Nowadays, Compton effect is applied in many other fields of science. One of its applications is the laser detector used in the research of spectra of plasma interactions. It also facilitates the scientific research in astrophysics, medical imaging and visualization of radioactive substances.

References

[1] Thomson, J. J. Conduction of Electricity through Gases. Cambridge University Press, 1906.

[2] Chen, Sy., Maksimchuk, A. Umstadter, D. Experimental observation of relativistic nonlinear Thomson scattering. Nature, 1998, 396: 653-655.

[3] Compton, A. H. A Quantum Theory of the Scattering of X-rays by Light Elements. Phys. Rev., 1923, 21: 483.

[4] Qiao C.-K., Wei J.-W., Chen L. An Overview of the Compton Scattering Calculation. Crystals 2021, 11: 525.

[5] Hamouda, Samir A. Compton Scattering: A Theory and Experiments. International Journal of Geology, Agriculture and Environmental Sciences, 2017, 5(4): 20-27.

[6] Klein O., Nishina Y. Über die Streuung von Strahlung durch freie Elektronen nach der neuen relativistischen Quantendynamik von Dirac. Z. Phys. 1929, 52: 853-868.

[7] Dumond, J. W. M. Compton Modifified Line Structure and its Relation to the Electron Theory of Solid Bodies. Phys. Rev. 1929, 33: 643-658.

[8] Eisenberger P., Reed W. A. Relationship of the relativistic Compton cross section to the electron's velocity distribution. Phys. Rev. B 1974, 9: 3237-3241.

[9] Tashima H., Yamaya T. Compton imaging for medical applications. Radiol Phys Technol, 2022, 15: 187-205.

[10] Webber J., Miller, E. L. Compton scattering tomography in translational geometries. Inverse Problems, 2020, 36(2): 025007.

[11] Chew C. R. K. Error in derivation of compton scattering formula, vixra:2010.0208, 2020.

[12] Falkenburg B. Particle metaphysics: a critical account of subatomic reality. Springer, 2010.

[13] Griffiths David J., Schroeter Darrell F. Introduction to Quantum Mechanics. Cambridge University Press, 2018.

[14] Sakurai J. J. Advanced Quantum Mechanics; Addison-Wesley: New York, NY, USA, 1967.

[15] Taylor J. R., Zafiratos C. D., Dubson M. A. Modern Physics for Scientists and Engineers (2nd ed.), 2020.

[16] Zheng Y., Xiang L. Study on Collision Process between Electron and Photon in Compton Effect, Physics and Engineering, 2013, 23(3): 18-20.