Planck Formula for Black-body Radiation: Derivation and Applications

Xiaoyang Guo[1, *, †] and Xinzhi Xie[2, †]

[1]Department of Science, Beijing University of Technology, Beijing, China

[2]Shandong Experimental High School, Jinan, China

†These authors contributed equally.

*Corresponding author: guoxiaoyang@emails.bjut.edu.cn

Abstract. At the end of the 19th century, German physicist Victor Brooks first proposed black hole radiation. He found that objects emit electromagnetic waves when they are heated. Brooks found that the spectral density of electromagnetic waves is proportional to the temperature of the body, which is called blackbody radiation. The research of blackbody radiation began in the middle of the 19th century, studying radiator and radiation standards to meet the needs of industrial applications. The experimental results show that, especially by interpreting the spectral distribution of Planck's formula, many new physics and concepts have emerged, and many important material concepts, quantum mechanics, quantum theory of solid conductors, induction emission, quantum statistics, etc. have been developed. This study enriches thermodynamics and provides insights into the nature of light. The first aim of this paper is to derive the Planck’s Law using two different methods: one is by following the Planck’s derivation, while the other is by using Planck’s interpretation of his formula. The second aim of this paper is to use the Planck’s Law to derive Wien’s Law and Stefan’s Law.

Keywords: Plank’s formula; Black-body radiation; Wien’s displacement law; Stefan’s law.

1. Introduction

Blackbody radiation refers to the electromagnetic waves emitted by objects when they receive thermal energy. This electromagnetic wave is generated by thermodynamic processes on the surface of an object. Blackbody radiation is a random process characterized by a spectral density that is proportional to the temperature of the object [1]. This interesting field of physics is the research topic of this article.

Blackbody radiation is the cradle of modern physics, which has created new content in physics. They have greatly promoted the progress of human science, including quantum mechanics, solid state quantum theory, stimulated radiation, quantum statistics, relativistic statistics, and so on. This has always been an important topic in physical research. As early as 1900, the German scientist Planck established the formula for blackbody radiation. In the next century, many scientists, such as Debye, Allen Fist, Laue, Lorenz, Poincare, Pauli, Bose, and Einstein, have made countless efforts for his research. To this day, black body radiation is still an important topic for human development. At the same time, in the development of modern physics, many physical formulas and theorems related to blackbody radiation are also being developed, such as Planck's formula, Wien's displacement law, and Stephen's law [2].

2. Derivation of Planck’s Formula

2.1. Number of Modes

In theory, spectral energy density of the black body can be calculated as the product of the average energy per mode and the number of modes per unit volume in interval $[\nu, \nu + d\nu]$. The number of modes per unit volume in interval $[\nu,\ \nu + d\nu]$ can be calculated using the number of modes below frequency ν. Now consider the standing wave in a cubic box with edge length L. The allowed modes are [3]:

$$A\sin\left(\frac{n_x\pi}{L}x+\frac{n_y\pi}{L}y+\frac{n_z\pi}{L}z\right) \tag{1}$$

Where$n_x, n_y, n_z \in N$. Define $n_r = \sqrt{{n_x}^2+{n_y}^2+{n_z}^2}$, then the frequency of a specific mode (n_x, n_y, n_z) will be $\nu_r = \frac{v}{\lambda_r} = \frac{n_r v}{2L}$. Thus, for number of modes that have frequency less than ν is the same as the number of modes that have $n_r < \frac{2\nu L}{v}$. Denote $\frac{2\nu L}{v} = n$ and when n is large enough, the number of modes that have frequency less than ν can be approximated by an integral:

$$N(\nu) = \iiint dn_x dn_y dn_z = \int_0^{n_r}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} {n_r}^2 \sin\theta \, d\varphi d\theta dn_r = \frac{4\pi}{3}\frac{L^3}{v^3}\nu^3. \tag{2}$$

Here, $L^3(=V)$ is the volume of the system, and for the problem this paper is trying to solve, the speed of wave $v = c$. Thus equation Eq. (2) becomes $N(\nu) = \frac{4\pi}{3}\frac{\nu^3}{c^3}V$. The number of modes per unit volume in interval [ν, ν + dν] is $D_0(\nu) = \frac{1}{V}\frac{dN}{d\nu} = 4\pi\frac{\nu^2}{c^3}$. However, it still needs to take account of the polarization, and the actual result is [4]

$$D(\nu) = 2D_0(\nu) = 8\pi\frac{\nu^2}{c^3} \tag{3}$$

2.2. Average Energy for Each Mode

2.2.1 Classical Approach

The fundamental thermodynamic relation states that

$$dU = TdS - PdV \tag{4}$$

For oscillator, $dV = 0$. Thus, the fundamental thermodynamic relation becomes $\frac{dS}{dU} = \frac{1}{T}$. With Wien's law, it is found that [5]

$$B_\nu(T) = \frac{8\pi\nu^2}{c^3}\mathrm{U} = C_1\nu^3 e^{-\frac{b_1\nu}{T}}. \tag{5}$$

The average energy for each oscillator: $U = a_1\nu e^{-\frac{b_1\nu}{T}}$. Rewriting it gives the following relation $\frac{1}{T} = -\frac{1}{b_1\nu}\ln\frac{U}{a_1\nu} = \frac{dS}{dU}$. Taking the second derivative, it yields $\frac{d^2S}{dU^2} = -\frac{1}{b_1\nu U}$. Denote $R = \left(\frac{d^2S}{dU^2}\right)^{-1}$, for Wien's Law $R_1 = A_1\mathrm{U}$, and with Rayleigh's law

$$B_\nu(T) = \frac{8\pi\nu^2}{c^3}U = C_2\nu^2 T. \tag{6}$$

Following similar procedure, it yields $R_2 = A_2U^2$. Due to the fact that Wien's law works well for high frequency and Rayleigh's law works well for low frequency, it is natural to assume the actual R is the linear combination of R_1 and R_2, i.e.,

$$R = AU + \mathrm{B}U^2 = \frac{1}{\frac{d^2S}{dU^2}}, \tag{7}$$

Which leads to:

$$\frac{dS}{dU} = \int\frac{1}{AU+BU^2}dU = \frac{1}{A}\ln\frac{U}{A+BU} + \ln C = \frac{1}{T}. \tag{8}$$

Eventually it gives the relation

$$U = \frac{AC}{e^{-\frac{A}{T}} - BC}, \tag{9}$$

Where A, B and C are constants, which can be obtained from experiments.

2.2.2 Quantum Approach

According to Plank's interpretation, an oscillator with frequency ν have quantized energy as follow: $\varepsilon_n = nh\nu$. Denote $\beta = 1/k_B T$, where k_B is the Boltzmann constant and T is the temperature of the system. The partition function for such oscillator is

$$Z = \sum_{n=0}^{\infty} e^{-\beta\varepsilon_n} = \sum_{n=0}^{\infty} e^{-\beta nh\nu} = \sum_{n=0}^{\infty} \left(e^{-\beta h\nu}\right)^n. \tag{10}$$

With $-\beta h\nu < 0$, which leads to $0 < e^{-\beta h\nu} < 1$, the series converge into $Z = \frac{1}{1-e^{-\beta h\nu}}$. The average energy for the oscillator will be [6]

$$U(\nu) = \langle\varepsilon\rangle = -\frac{\partial lnZ}{\partial\beta} = \frac{h\nu}{e^{\beta h\nu} - 1}. \tag{11}$$

2.3. Result

Combining Eq. (3) and Eq. (11), it gives that

$$B_\nu(T) = U(\nu)D(\nu) = \frac{8\pi\nu^2}{c^3}\frac{h\nu}{e^{\beta h\nu} - 1}. \tag{12}$$

This is the spectral energy of the black body radiation in terms of frequency. The spectral energy in terms of wavelength can be derive from the relation

$$B_\lambda(\mathrm{T})\mathrm{d}\lambda = -B_\nu(\mathrm{T})\mathrm{d}\nu. \tag{13}$$

Here, minus sign is due to the fact that increase in wavelength leads to the decrease in frequency. With the relation between wavelength and frequency $\nu = c/\lambda$ given, it gives out the result [7]

$$B_\lambda(\mathrm{T}) = -B_\nu(\mathrm{T})\frac{d\nu}{d\lambda} = \frac{8\pi h}{\lambda^3}\frac{1}{e^{\frac{\beta hc}{\lambda}} - 1}\frac{c}{\lambda^2} = \frac{8\pi hc}{\lambda^5}\cdot\frac{1}{e^{\frac{\beta hc}{\lambda}} - 1}. \tag{14}$$

The intensity-wavelength curves for blackbody radiation spectrum are shown in Fig. 1.

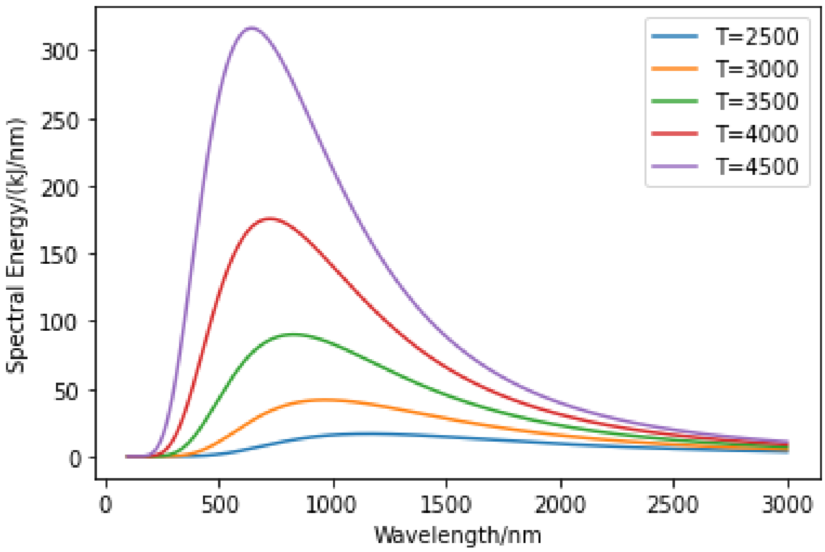

Fig. 1 The intensity-wavelength curves for blackbody radiation spectrum

3. Applications

3.1. Wien's Displacement Law

In consistent with Eq. (11), the blackbody radiant energy flux density spectrum in terms of the wavelength is

$$\mathrm{u}(\lambda)=\frac{8\pi hc}{\lambda^5}\cdot\frac{1}{e^{\frac{hc}{\lambda kT}}-1}. \tag{15}$$

Then differentiate the $u(\lambda)$ to get the value of λ when u gets to the maximum, the result is

$$\frac{\partial U}{\partial\lambda}=8\pi hc\left(\frac{hc}{kT\lambda^7}\frac{e^{\frac{hc}{\lambda kT}}}{(e^{\frac{hc}{\lambda kT}}-1)^2}-\frac{1}{\lambda^6}\frac{5}{e^{\frac{hc}{\lambda kT}}-1}\right), \tag{16}$$

Which is equivalent to the equation $\frac{hc}{\lambda kT}\frac{e^{\frac{hc}{\lambda kT}}}{e^{\frac{hc}{\lambda kT}}-1}-5=0$. In order to facilate the calculation, it is helpful to denote $\frac{hc}{\lambda kT}=\alpha$ and thus it is found that

$$5(\mathrm{e}^{\alpha}-1)=\alpha\mathrm{e}^{\alpha}. \tag{17}$$

There are many methods, including image method, iteration method and other approximate solutions, can be used to solve this transcendental equation [8]. The solution is $\alpha=4.965114231744278$. Puting it back, it is arrived that [9]

$$\lambda_{\max}T=\frac{hc}{k\alpha}=2.8977721\times10^6\mathrm{K}\cdot\mathrm{nm}, \tag{18}$$

Which is the so-called Wien's displacement law.

3.2. Stefan's Law

According to Planck's formula which states that

$$M_B(\lambda,T)=2\pi c^2\lambda^{-5}\frac{1}{e^{\frac{hc}{k\lambda T}}-1}. \tag{19}$$

To make the derivation visually simple, the following substitutions $C_1=2\pi c^2$ and $x=\frac{hc}{k\lambda T}$ are made. Thus the wavelength λ can be write into the form $\lambda=\frac{hc}{kTx}$, which means that $d\lambda=-\frac{hc}{kTx^2}dx$. At the same time, by bringing x back to Planck's formula, Eq. (19) can be rewritten as

$$M_B(\lambda,T)=\frac{C_1\lambda^{-5}}{e^x-1}=\frac{C_1x^5}{x^5\lambda^5}\cdot\frac{1}{e^x-1}=\frac{C_1k^5T^5}{h^5c^5}\cdot\frac{x^5}{e^x-1}=M_B(x,T). \tag{20}$$

Therefore, the radiating degree of blackbody at temperature T is

$$M_B(\mathrm{T})=\int_0^\infty M_B(\lambda,\mathrm{T})\mathrm{d}\lambda=\frac{C_1k^4T^4}{h^4c^4}\int_0^\infty\frac{x^3}{e^x-1}\mathrm{d}x. \tag{21}$$

By using of the fact that $\int_0^\infty\frac{x^3}{e^x-1}dx=6.494$, it is easy to find that [10]

$$M_B(T)=6.494\frac{C_1K^4}{h^4c^4}T^4=\sigma T^4 \tag{22}$$

where $\sigma=6.494\frac{C_1K^4}{h^4c^4}T^4=5.710\times10^{-8}\mathrm{W}/(\mathrm{m}^2\cdot\mathrm{k}^4)$. The above equation is known as the Stefan's law.

4. Conclusion

In this paper, black-body radiation has been deeply studied with various mathematical methods and physical thinking processes. Throughout the paper, two derivation methods of Planck's formula are introduced, and based on this, a further exploration is launched. This paper also demonstrated Wien's displacement law and Stephen's law, and during this period a large number of proof methods were used. Further deepened the understanding of black-body radiation and quantum physics, and further proved it with image method. It is easy to understand by combining numbers with shapes. Black-body radiation and its derived conclusions have played an irreplaceable role in human life. Whether it is high-end technology or daily household products, the application of black-body radiation has been fully utilized all over the world and has become an integral part of human life. Black-body radiation is widely used in daily applications, such as water heaters, hot ovens, ovens and other electrical appliances. In addition, black-body radiation has important applications in industry, metallurgy, chemical industry, construction and other fields. In the research, black-body radiation is also used to study the structure of matter and thermodynamic process. In addition, black-body radiation is also an important part of quantum physics. A better grasp of black-body radiation will further physics. The research and exploration of black-body radiation continue, and there will be more discoveries and inventions in the future.

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