

Exact and Approximate Formulae for the Period of Simple Pendulum

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Abstract. Pendulum plays a significant role in certain practical aspects such as measuring gravitational acceleration and proving the rotation of Earth. In traditional textbooks, only a simplified approximate formula for calculating pendulum period in small angle is given. As a matter of fact, the analytical derivation of the pendulum period can always be given in the form of elliptical integral simply using the law of the conservation of energy. The form, however, is relatively complex for students and beginners in physics. Thus, a simplified version of the period of simple pendulum is highly desirable. This article is going to present a series of approximate formulae for pendulum period and conduct some comparison among these different formulae. This essay is dedicated to help readers and beginners in physics to look into the process of deriving the pendulum period formula, and also provide a convenient way for readers to look up certain literatures in this research area.

Keywords: Simple pendulum, Analytical formula, Approximation, Taylor series.

1. Introduction

A simple pendulum is a device capable of producing reciprocating swings. Usually, a simple pendulum consists of a weightless (or very light) thin rope that is not extensible and a massive bob. One end of the rope is hanged to a fixed point in the gravity field, and the other end is consolidated with the massive bob. This device, having been studied since 1700s, contribute to the study of Physics in certain aspects for it can used to prove the rotation of the Earth and measure the acceleration of the gravity [1].

The pendulum period formula recorded in textbooks is just a rough approximate result that only produces precise data in small angle. The analytical derivation of the pendulum period T can be easily given by using the law of the conservation of energy in the form of elliptical integral. There has already been a variety of approximate formula given in numerous studies. For example, Belendez and colleagues published a study that gives the approximated formula using Taylor series [2], and the study by Belendez et al. claim their formula produces relative error as low as 0.013% at an angle of 140° [3]. Another study gives an approximated formula that is precise when the starting angle is smaller than 15° by using graph and function fitting [4].

This paper is dedicated to help readers and beginners in physics to look into the formula of calculating pendulum period while also provide a convenient way for readers to look up certain literatures. Section 2 presents the derivation of the pendulum period formula recorded in textbooks $T = 2\pi\sqrt{l/g}$ in natural system of coordinates. Section 3 includes three parts. Sec. 3.1 presents the process of deriving the pendulum period into the form of elliptical integral, Sec. 3.2 shows the approximation of the elliptical integral using Taylor series by this article, and Sec. 3.3 records some other studies that gives its own approximated formula for the pendulum period and the principle they adopted. The last section is devoted to the conclusion.

2. Derivation of the Simplified Pendulum Period Formula

Under the circumstances of the natural system of coordinates, the vector of a unit varies in different condition. Based on Fig. 1, the pendulum is now swaying from O' to A' . $\vec{e}_n$ is the normal vector, and $\vec{e}_r$ is the unit tangent vector. P is a random position on the trajectory of the pendulum ball

swaying from O' to A' . Noting the velocity of the pendulum ball at P as v , the Kinetic equation in the direction of $\vec{e}_r$ can be expressed as [5]

$$-mg \sin \theta = m \frac{dv}{dt}. \quad (1)$$

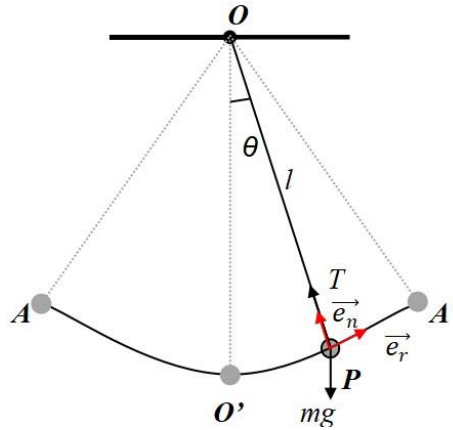


Fig. 1 The schematic diagram of the pendulum model

Since $v = \frac{d\theta}{dt} l$ and the relation $\sin \theta \approx \theta$ holds in small angle, Eq. (1) can also be expressed as

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0. \quad (2)$$

Also, noting that $\omega^2 = \frac{g}{l}$, the second-order linear differential equation can be solved to get the period of the pendulum

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{l}{g}}. \quad (3)$$

The Eq. (3) is solved under the condition of swaying from O' to A' , it can also be solved under other conditions such as swaying from O' to A or A to O' using similar methods and principles.

3. Analytic Solution for the Pendulum Period Formula

3.1. Analytic Derivation

The Fig. 1 shows a pendulum model where the quality of the pendulum ball is noted as m , the length of the string is noted as r , and the vertical angular separation during the swing of the sting is noted as θ . The equation for the swing period T of the pendulum can be derived as the following [6]. As the energy lost due to non-conservative forces such as air resistance is ignored, the total mechanical energy during the swing is conserved. Therefore,

$$\frac{1}{2}mv^2 + mgr(1 - \cos \theta) = E. \quad (4)$$

The lowest point during the swing is taken as the potential zero, where, as the equilibrium position, the potential energy is zero. θ_0 is defined as the largest value of θ . When $\theta_0 = \theta$, the kinetic energy of the pendulum equals zero, and the total mechanic equals the potential energy alone. The equation of the total mechanic energy can be expressed as

$$E = mgr(1 - \cos \theta_0). \quad (5)$$

The velocity of the pendulum can be expressed by uniting Eq. (4) and Eq. (5) simultaneously as the equation below

$$v = \sqrt{2gr(\cos \theta - \cos \theta_0)}, \quad (6)$$

which can also be expressed as $v = \omega r = \frac{d\theta}{dt} \times r$. Substituting this relation into Eq. (6), a simplified equation can be derived as

$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{r} (\cos \theta - \cos \theta_0)}. \quad (7)$$

Since $\cos \theta = 1 - 2\sin^2 \frac{\theta}{2}$ and $\cos \theta_0 = 1 - 2\sin^2 \frac{\theta_0}{2}$, it is observed that the derivative can be rewritten as $\frac{d\theta}{dt} = 2\sqrt{\frac{g}{r} (\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2})}$. The period which takes for the pendulum to sway from θ to the potential zero can be expressed as

$$T = \frac{1}{2} \sqrt{\frac{r}{g}} \int_0^{\theta} \frac{d\theta}{\sqrt{\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2}}}. \quad (8)$$

Because θ is just a random vertical angular separation from 0 to θ_0 , the following transformation can be made: $\sin^2 \frac{\theta}{2} = \sin^2 \frac{\theta_0}{2} \sin^2 \varphi$. Taking the differential of each side of the above equation, it is inferred that $\frac{1}{2} \cos \frac{\theta}{2} d\theta = \sin^2 \frac{\theta_0}{2} \cos \varphi d\varphi$. Thus Eq. (8) can be transformed into

$$T = \frac{1}{2} \sqrt{\frac{r}{g}} \int_0^{\varphi} \frac{2 \sin^2 \frac{\theta_0}{2} \cos \varphi / \cos \frac{\theta}{2}}{\sqrt{\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta_0}{2} \sin^2 \varphi}} d\varphi = \sqrt{\frac{r}{g}} \int_0^{\varphi} \frac{d\varphi}{\sqrt{1 - \sin^2 \frac{\theta_0}{2} \sin^2 \varphi}}. \quad (9)$$

3.2. Approximate Formulae

Using the analytical equation derived in Sec. 3.1, an approximated formula can actually be derived as following. Noticing that in Eq. (8) the relation $\sin^2 \frac{\theta}{2} = \sin^2 \frac{\theta_0}{2} \sin^2 \varphi$ holds, where φ has a value of 0 when θ equals 0 and φ has a value of $\frac{\pi}{2}$ when θ equals the largest angle θ_0 [7]. Thus, $\varphi \in [0, \frac{\pi}{2}]$ and the pendulum period T can be expressed as

$$T = 4 \sqrt{\frac{r}{g}} \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1 - \sin^2 \frac{\theta_0}{2} \sin^2 \varphi}}. \quad (10)$$

When θ_0 is very small, $\sin^2 \frac{\theta_0}{2} \rightarrow 0$, $\sqrt{1 - \sin^2 \frac{\theta_0}{2} \sin^2 \varphi} \rightarrow 1$, and the pendulum period T can be expressed as $T \approx 4 \sqrt{\frac{r}{g}} \int_0^{\frac{\pi}{2}} d\varphi = 2\pi \sqrt{\frac{r}{g}}$, which is merely Eq. (3). However, this approximate formula can only produce precise result in small starting angle (usually smaller than 5 degree). Thus, by using the Maclaurin-Taylor series, a more accurate approximated formula can be derived.

In step Eq. (8), $(\cos \theta - \cos \theta_0)$ is transformed via two-fold duplication formula. To get a more precise formula that can be used to calculate the pendulum period in an even bigger angle, another derivation can be made using the Maclaurin-Taylor series [8], i.e.,

$$dt = \sqrt{\frac{r}{2g}} d\theta (\cos \theta - \cos \theta_0)^{-\frac{1}{2}}. \quad (11)$$

According to the Taylor series, it is observed that $\cos \theta = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} = 1 - \frac{1}{2} \theta^2 + \frac{1}{24} \theta^4 + \dots$ and $\cos \theta_0 = \sum_{n=0}^{\infty} \frac{(-1)^n \theta_0^{2n}}{(2n)!} = 1 - \frac{1}{2} \theta_0^2 + \frac{1}{24} \theta_0^4 + \dots$. By taking the first three terms of $\cos \theta$ and $\cos \theta_0$ using the Taylor series, Eq. (11) can be transformed into

$$dt \approx \sqrt{\frac{r}{2g}} d\theta \left[\frac{1}{2} (\theta_0^2 - \theta^2) - \frac{1}{24} (\theta_0^4 - \theta^4) \right]^{-\frac{1}{2}} = \sqrt{\frac{r}{2g}} d\theta \frac{\sqrt{2}}{\sqrt{\theta_0^2 - \theta^2}} \left(1 - \frac{\theta_0^2 + \theta^2}{12} \right)^{-\frac{1}{2}}. \quad (12)$$

The equation above can be approximated using binomial expansion:

$$dt = \sqrt{\frac{r}{g}} \frac{d\theta}{\sqrt{\theta_0^2 - \theta^2}} \left(1 + \frac{\theta_0^2 + \theta^2}{24} \right). \quad (13)$$

Taking the integral of both sides of the equation, the pendulum period T can be expressed as

$$T \approx 2 \sqrt{\frac{r}{g}} \int_{-\theta_0}^{\theta_0} \frac{d\theta}{\sqrt{\theta_0^2 - \theta^2}} \left(1 + \frac{\theta_0^2 + \theta^2}{24} \right). \quad (14)$$

Let θ equals to $\theta_0 \sin t$, then the period can be recast into

$$T \approx 2 \sqrt{\frac{r}{g}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1 + \frac{\theta_0^2 + \theta^2}{24} \right) \frac{\theta_0 \cos t dt}{\theta_0 \cos t} = 2 \sqrt{\frac{r}{g}} \left[t + \frac{\theta_0^2}{16} t - \frac{\theta_0^2}{96} \sin 2t \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}. \quad (15)$$

After the equation above is simplified and arranged, a more precise formula for calculating the period T of the pendulum is derived as

$$T \approx 2\pi \sqrt{\frac{r}{g}} \left(1 + \frac{\theta_0^2}{16} \right). \quad (16)$$

3.3. Other High-Accuracy Pendulum Period Formulae

Though a precise approximate formula has already been given by this article, formulae derived in accordance with other mathematical principles should also be considered when a lower error result is expected to be produced. The first example is

$$\frac{T}{T_0} \approx 1 - \frac{2}{\pi} \ln \cos \frac{a}{2} - 0.1684 \left(\frac{a}{180} \right)^2 \sqrt{1 - 0.51 \left(\frac{a}{180} \right)^3}. \quad (17)$$

Formula Eq. (17) can be derived by combining the use of inequality and power series together according to a study [2]. The absolute error of Eq. (17) is claimed to be less than 0.0006, and the relative error is claimed to be less than 0.05% in the amplitude range close to 180 degree by the study. The second example is

$$\frac{T_{a2}}{T_0} = 1 + \frac{1}{16} \theta_0^2 + \frac{11}{3072} \theta_0^4 + \frac{173}{737280} \theta_0^6 + \frac{22931}{1321205769} \theta_0^8 + \dots, \quad (18)$$

Which is similar to the one derived in the article by using the Taylor series [3]. This study claimed that it used power series instead of Taylor series when deriving the approximated formula from the basic elliptical integral. The third example is

$$T = T_0 \frac{\varphi_0 + 0.082\varphi_0^3 + 0.032\varphi_0^5}{\sin \varphi_0}. \quad (19)$$

This formula is given by using graph and function fitting to manifest the basic elliptical integral function [4]. By developing the correction formula and expand the elliptical integral derivation, the fourth example is

$$T \cong 2\pi \sqrt{\frac{l}{g} \cos \frac{a}{2}} \quad (20)$$

for $a \leq \frac{\pi}{2}$ [9]. Based on the conclusion of a former study, another study further expands the formula derived by Kidd and Fogg using power series [10], the fifth example is

$$\frac{T_{app1}}{T_0} = 1 + \frac{1}{16} \theta_0^2 + \frac{14}{3072} \theta_0^4 + \frac{278}{737280} \theta_0^6 + \dots \quad (21)$$

4. Conclusion

The simple pendulum serves as a workhorse in college physics and other related topics. It has many applications in several disciplines and its period is identified to have an exact solution via the elliptical integral. This paper aims to study the period of the simple pendulum by using both analytical approaches and approximate tricks. The main purpose of this essay is to give a simpler interpretation about the complicated elliptical integral form of the analytical pendulum period formula, and at the same time provide a variety of other studies that also give their own approximate formulae. This article presents the main mathematical details and displays the principles adopted to derive approximate formulas. With this paper, readers can be offered with an accessible way to look into the process of deriving the pendulum period formula and get some insight about it. What's more, this essay can also be used by students who also want to conduct their own study to consult certain literatures about deriving approximate pendulum formulae and compare their pros and cons. Nevertheless, this work inevitably has some shortcomings since the approximate expression of the elliptical integral is extremely hard to find. Therefore, a Padé approximant approach can be used to hunt for a more efficient formula in the future.

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