

Applications of Cauchy's Residue Theorem in Computing Improper Integral

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Abstract. An improper integral is a definite integral that either has an infinite interval or has the integrand that is not defined on some points in the interval. Many improper integrals are difficult to compute by using real analysis methods, especially those containing infinity. By contrast, introducing the complex methods and applying Cauchy's residue theorem can give a much more simplified solution. In order to apply Cauchy's residue theorem, the residues of the integrand at the singularities that are interior to the contour are first to be found, then the integral along the whole simple closed contour can be evaluated. These contours always consist of line segments and sectors of circle. In most cases, only the part of contour on the real axis is related to the real definite integral, and other parts should be eliminated by proving its tendency to a certain value where most commonly it is zero. This is always done by considering the property of the integrand or using Jordan lemma.

Keywords: Improper integrals, Cauchy's residue theorem, Definite integrals, Calculus.

1. Introduction

Mainly developed in the 19th century, complex analysis was inaugurated by the French mathematician Augustin-Louis Cauchy, who was inspired by the potential equation and celestial mechanics in physics. It is then fully established along with the invention of Cauchy's integral formula and Cauchy's residue theorem [1]. These techniques enable more kinds of differential equations to be solved and integral to be computed. After that, both Abel and Jacobi discovered elliptic functions, independently, which contribute greatly to the development of the theory of complex functions [2]. Cauchy's work was further developed by Bernhard Riemann and Karl Theodor Wilhelm Weierstrass, as well as Goursat Eouard-Jean-Baptiste [3]. Riemann introduced the definition of Riemann surfaces, and later, holomorphic functions. Complex analysis was applied in various physical fields, such as gravitation and electrostatics, in the 19th century, and later in fluid dynamics, elasticity and string theory [4].

Cauchy's Residue theorem has applications in various fields. In pure mathematics, for instance, it can be used to prove the d'Alembert's theorem [5], evaluate the sum of infinite series [6], and compute improper integral. In physics, Cauchy's residue theorem can be applied to analyze physical models, such as modeling the process of heat conduction [7].

In this paper, basic concepts and theorems will be introduced in Section 2. In Section 3, these methods would be applied on four examples, and all improper integrals would be evaluated with the aid of properly chosen contours. It might be difficult or even impossible to integrate them using the real method, since they involve integrating from negative infinity to infinity, or from zero to infinity. However, by using Cauchy's residue theorem, this problem can be avoided. Jordan lemma and other expressions are also used to evaluate the integral of the function on an arc that is part of the contour. In example 1 and example 2, the contours are chosen so that singular points are interior to the contour as these singularities do not fall on the real axis. Nevertheless, in example 3 and example 4, an intended path is created to avoid the singular points on the real axis.

2. Methods and Theorems

2.1. Cauchy-Goursat Theorem

Cauchy-Goursat theorem lays the foundation of the proof, which states that if a function f is analytic at all points interior to and on a simple close contour, then the integral of this function along that contour equals to zero. $\int_C f(z)dz = 0$. Another useful theorem is Cauchy integral formula and its extension. It is said that if f is analytic both inside and on a simple closed contour C , and z_0 is any point inside C , it follows that

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{z-z_0} \quad (1)$$

In fact, Eq. (1) can be extended to give expression for any order of derivative [8]

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)dz}{(z-z_0)^{n+1}}. \quad (2)$$

If a function $f(z)$ is analytic throughout $N_r(z_0)$, then there is a representation for $f(z)$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad (3)$$

Where $a_n = \frac{f^{(n)}(z_0)}{n!}$. Obviously, this is the Taylor's expansion of function $f(z)$ at z_0 .

If a function $f(z)$ is analytic throughout a domain with z satisfying $R_1 < |z - z_0| < R_2$, and that C is a simple closed contour lying in the domain, then $f(z)$ has the expression

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}, \quad (4)$$

And that $a_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z-z_0)^{n+1}}$ and $b_n = \int_C \frac{f(z)dz}{(z-z_0)^{-n+1}}$. This is the Laurent's expansion of $f(z)$ at z_0 .

2.2. Residue Theorem

If a function $f(z)$ is analytic both inside and on a simple closed contour C , with a finite number of isolated singularities, then its integral around C equals to $2\pi i$ multiplies the sum of the residues of the function at the singularities that are interior to contour C , i.e.,

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^n \text{Res}(f(z), z_k). \quad (5)$$

Note that for every a_n , the expression in Laurent's series is identical with that in Taylor series. Thus, it is sensible to regard Laurent series as an extension of Taylor series, and the negative power terms are included [9]. Consider the term b_1 , which is equals to $\int_C f(z) dz$. This is a special term, since the integral of $f(z)$ along the contour can be obtained by computing b_1 . The residue of function f at the point z_0 is denoted by $\text{Res}(f(z), z_0)$, were

$$\text{Res}(f(z), z_0) = b_1 = \frac{1}{2\pi i} \int_C f(z) dz. \quad (6)$$

To prove the Cauchy's Residue theorem, it is started by creating circles centered at each isolated singular points z_k with sufficiently small radius that allows each circle C_k to be contained in the contour C and do not intersect each other. Since there are finite number of isolated singular points, this can be achieved. Then, one can use line segments to connect all the circles and the contour C to

generate two simple closed contours. Since the singular points are omitted, $f(z)$ is analytic inside and on these two new contours. By applying Cauchy-Goursat theorem, integral along these two contours equals to zero, and this result can also be expressed by

$$\int_C f(z) dz - \sum_{k=1}^n \int_{C_k} f(z) dz = 0. \quad (7)$$

This is reasonable as the integral along curves used for connection cancels out. Rearrange the expression and use definition of residue, it is inferred that $\int_{C_k} f(z) dz = 2\pi i \operatorname{Res}(f(z), z_k)$ for each C_k , and thus $\int_C f(z) dz$ is equal to the sum of residues at each singular points.

2.3. Methods to Find Residues

If the coefficients of Laurent series of a function $f(z)$ at a singular point satisfies that $b_n \neq 0$ and that $b_k = 0$ for all $k \geq n + 1$, then the singular point is defined as a pole of order n . If z_0 is a pole of order n of f , then f can be expressed in the form

$$f(z) = \frac{g(z)}{(z-z_0)^n}, \quad (8)$$

Where $g(z)$ is analytic and non-zero at z_0 , and that

$$\operatorname{Res}(f(z), z_0) = \frac{g^{(n-1)}(z)}{(n-1)!}, \quad (9)$$

For all interger $n \geq 1$.

It is proceeded to consider all the terms in Laurent series of $f(z)$. multiplies each term by $(z - z_0)^n$, an infinite polynomial series with respect to $(z - z_0)$ can be obtained, which is analytic in the whole complex plane. This is the function $g(z)$ required. Furthermore, as $g(z)$ is analytic, then $g(z)$ must have a Taylor expansion at point z_0 . Then divides each term by $(z - z_0)^n$, which gives the Laurent series of $f(z)$ at z_0 . This time, the coefficient of term of power of -1 becomes $\frac{g^{(n-1)}(z)}{(n-1)!}$. According to definition of residue, this is the residue of f at z_0 .

If $f(z_0) = f^{(1)}(z_0) = f^{(2)}(z_0) = \dots = f^{(n-1)}(z_0) = 0$, and that $f^{(n)}(z_0) \neq 0$, then $f(z)$ can be expressed in the form: $(z - z_0)^n g(z)$, where $g(z)$ is analytic and non-zero at z_0 . Since f is analytic, then consider its Taylor series at point z_0 . The first n terms are all equals to zero, so $(z - z_0)$ could be taken out as a common factor.

Next, the method for computing the residue is shown below. If $f(z)$ and $g(z)$ are both analytic at z_0 , and that $f(z_0) \neq 0, g(z_0) = 0, g^{(1)}(z_0) \neq 0$, then

$$\operatorname{Res}\left(\frac{f(z)}{g(z)}, z_0\right) = \frac{f(z_0)}{g^{(1)}(z_0)}. \quad (10)$$

From what is stated above, $g(z) = (z - z_0)h(z)$, where $h(z)$ is analytic and non-zero at z_0 . So $\frac{f(z)}{g(z)} = \frac{f(z)}{(z-z_0)h(z)}$. Let $p(z) = \frac{f(z)}{h(z)}$, so $p(z)$ is analytic at z_0 and therefore it has Taylor series at z_0 . Hence $\frac{f(z)}{g(z)} = \frac{p(z_0) + p^{(1)}(z_0)(z-z_0) + \frac{p^{(2)}(z_0)}{2!}(z-z_0)^2 + \dots + \frac{p^{(n)}(z_0)}{n!}(z-z_0)^n + \dots}{z-z_0}$ so the residue is $p(z_0)$. Since $g(z) = (z - z_0)h(z)$ and $g^{(1)}(z) = h(z_0)$, the residue is equal to

$$p(z_0) = \frac{f(z_0)}{h(z_0)} = \frac{f(z_0)}{g^{(1)}(z_0)}. \quad (11)$$

2.4. Other Theorems

Jordan's lemma states that if f is continuous for all z that satisfies $R_0 \leq |z| < \infty, \text{Im}(z) \geq 0$, and that $\lim_{z \rightarrow \infty} f(z) = 0$, then for any $k > 0$,

$$\lim_{R \rightarrow \infty} \int_{C_R} e^{ikz} f(z) dz = 0, \quad (12)$$

Where C_R denotes the semi-circle $Re^{i\theta}$ ($0 \leq \theta \leq \pi, R \geq R_0$).

If function $f(z)$ is continuous on a region C , where $C = \{z: z = a + Re^{i\theta}, 0 < R \leq R_0, \theta_0 \leq \theta \leq \theta_0 + \alpha\}$, and that $\lim_{z \rightarrow a} (z - a) f(z) dz = L$, then

$$\lim_{R \rightarrow 0} \int_{C_R} f(z) dz = iL\alpha, \quad (13)$$

Where $C_R = \{z: z = a + Re^{i\theta}, \theta_0 \leq \theta \leq \theta_0 + \alpha\}$ and it is positively oriented.

3. Applications

3.1. Example 1

The first example considered is

$$I = P.V. \int_{-\infty}^{\infty} \frac{x^2}{(x^2+9)(x^2+1)} dx. \quad (14)$$

This integral can be computed by integrating it on the complex plane. Let $f(z) = \frac{z^2}{(z^2+9)(z^2+1)}$. The singular points are $\pm 3i$ and $\pm i$. By symmetry, one can only consider the singular points above x -axis. By using Eq.(11), $\text{Res}(f(z), 3i) = \frac{z^2}{(z^4+10z^2+9)(1)} \Big|_{z=3i} = -\frac{3}{16}i$, $\text{Res}(f(z), i) = \frac{1}{16}i$.

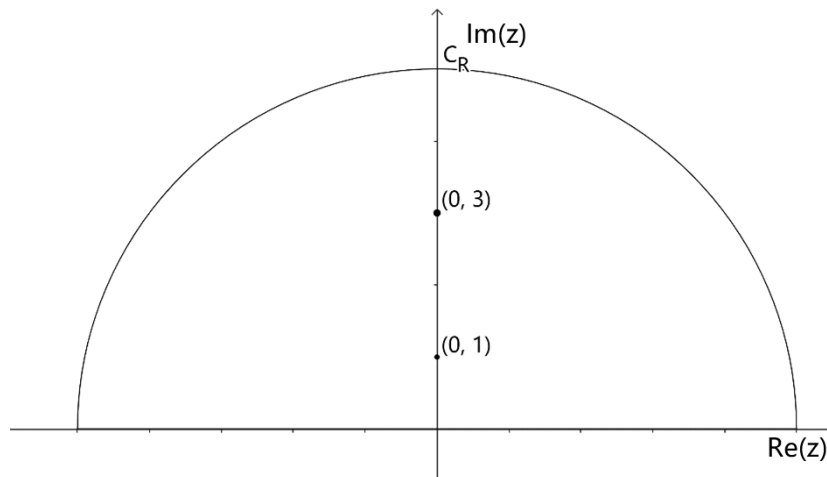


Figure 1. The contour of the integral in example 1

For convenient, a closed contour consisting of a line segment and a semi-circle of radius R is created, see Fig. 1. Here, $C = [-R, R] \cup C_R$, where C_R denotes the semi-circle. Only $3i$ and i are interior to the contour, so applying Cauchy's residue theorem,

$$\int_{-R}^R f(z) dz + \int_{C_R} f(z) dz = 2\pi i (\text{Res}(f(z), 3i) + \text{Res}(f(z), i)) = 2\pi i \left(-\frac{1}{8}i \right) = \frac{\pi}{4}, \quad (15)$$

Which is independent of R . $|f(z)| = \left| \frac{z^2}{(z^2+9)(z^2+1)} \right| = \frac{R^2}{|z^2+9||z^2+1|} \leq \frac{R^2}{||z^2|-9||z^2|-1|} = \frac{R^2}{|R^2-9||R^2-1|} = \frac{R^2}{(R^2-9)(R^2-1)}$. When $R \rightarrow +\infty$, the first term becomes $\int_{-\infty}^{\infty} f(x)dx$, whereas for the second term,

$$\left| \int_{C_R} f(z)dz \right| \leq \int_{C_R} |f(z)|dz \leq \pi R \frac{R^2}{(R^2-9)(R^2-1)} = \pi \frac{R^3}{(R^2-9)(R^2-1)}, \quad (16)$$

Which tends to zero. Therefore, the final result is

$$\lim_{R \rightarrow +\infty} P.V. \int_{-R}^R f(z) dz = \frac{\pi}{4}. \quad (17)$$

3.2. Example 2

The second example considered is [10]

$$I = P.V. \int_0^{\infty} \frac{1}{x^3+1} dx. \quad (18)$$

Let $f(z) = \frac{1}{z^3+1}$. The singular points are $e^{\frac{\pi}{3}i}$, $e^{\frac{2\pi}{3}i}$, -1 . Consider the contour shown in Fig. 2, and the contour C is consist of two line segments and an arc of $\frac{1}{3}\pi$ radian. The only singularity interior to C is $e^{\frac{\pi}{3}i}$, and $\text{Res}\left(f(z), e^{\frac{\pi}{3}i}\right) = \frac{1}{3(e^{\frac{\pi}{3}i})^2} = \frac{1}{3}e^{\frac{4}{3}\pi i}$. So applying Cauchy's residue theorem,

$$\int_0^R f(z) dz + \int_{C_R} f(z)dz + \int_R^0 f(re^{\frac{\pi}{3}i}) (re^{\frac{\pi}{3}i})^{(1)} dr = 2\pi i \cdot \frac{1}{3}e^{\frac{4}{3}\pi i} = \left(-\frac{1}{3}i + \frac{\sqrt{3}}{3}\right)\pi. \quad (19)$$

The left-hand side equals to $\int_0^R \frac{1}{z^3+1} dz - e^{\frac{2\pi}{3}i} \int_0^R \frac{1}{r^3+1} dr + \int_{C_R} \frac{1}{z^3+1} dz = \left(\frac{3}{2} - \frac{\sqrt{3}}{2}i\right) \int_0^R \frac{1}{z^3+1} dz + \int_{C_R} \frac{1}{z^3+1} dz$. Since $\left| \frac{1}{z^3+1} \right| \leq \frac{1}{||z^3|-1|} = \frac{1}{R^3-1}$, so as $R \rightarrow +\infty$, $\left| \int_{C_R} \frac{1}{z^3+1} dz \right| \leq \int_{C_R} \left| \frac{1}{z^3+1} \right| dz \leq \frac{1}{R^3-1} \cdot \frac{2}{3}\pi R = \frac{2}{3}\pi \frac{R}{1-\left(\frac{1}{R}\right)^3}$ tends to zero. Therefore, $\lim_{R \rightarrow \infty} \left(\frac{3}{2} - \frac{\sqrt{3}}{2}i\right) \int_0^R \frac{1}{z^3+1} dz = \left(-\frac{1}{3}i + \frac{\sqrt{3}}{3}\right)\pi$, and thus

$$\lim_{R \rightarrow \infty} \int_0^R \frac{1}{z^3+1} dz = \frac{2\pi}{3\sqrt{3}} \quad (20)$$

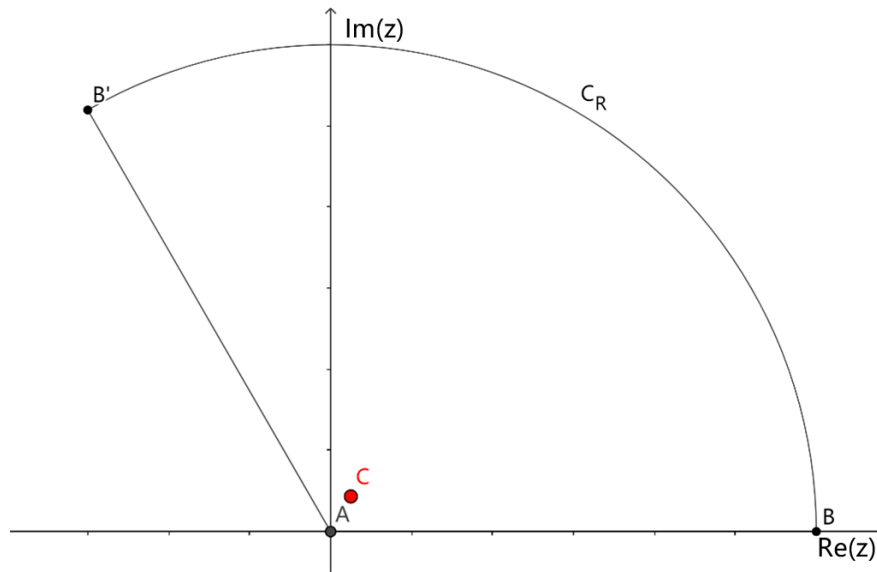


Figure 2. The contour of integral in example 2

3.3. Example 3

The third example considered is

$$I = \int_0^{+\infty} \frac{\sin(kx)}{x(x^2+a^2)} dx, \quad (21)$$

Where $k > 0$ and $a > 0$.

Let $f(z) = \frac{e^{ikz}}{z(z^2+a^2)}$ and consider the contour $C = C_R \cup [-R, -r] \cup C_r \cup [r, R]$, where C_R is positively oriented while C_r is negatively oriented, see Fig. 3. The only singular point interior to C is ia . By applying Eq.(11),

$$\text{Res}(f(z), ia) = \frac{e^{ik(ai)}}{3(ia)^2+a^2} = \frac{e^{-ka}}{-2a^2} = -\frac{e^{-ka}}{2a^2} \quad (22)$$

By virtue of Eq. (12) and Eq. (13), it is found that

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0 \quad (23)$$

And

$$\lim_{r \rightarrow 0} \int_{C_r} f(z) dz = i L\alpha = -i\pi \lim_{z \rightarrow 0} z f(z) = i\pi \frac{(ze^{ikz})^{(1)}}{[z(z^2+a^2)]^{(1)}} = -\frac{\pi i}{a^2}. \quad (24)$$

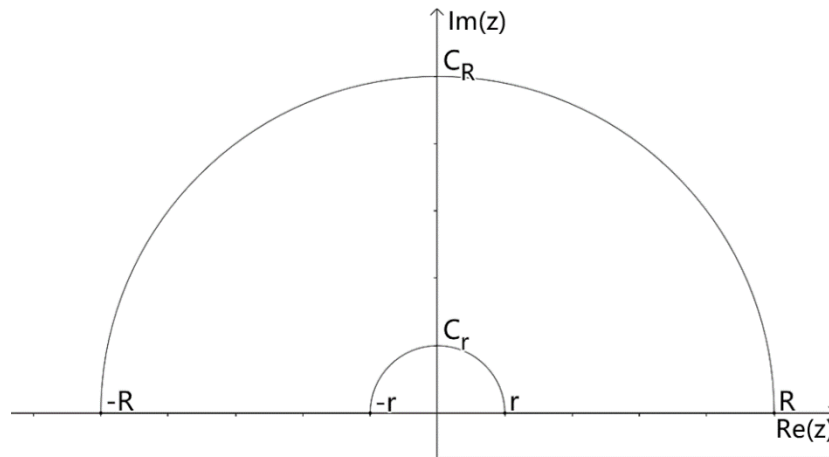


Figure 3. The contour of integral in example 3

In addition, by applying Cauchy's residue theorem, it is observed that

$$\int_{C_R} f(z)dz + \int_{-R}^{-r} f(z)dz + \int_{C_r} f(z)dz + \int_r^R f(z)dz = 2\pi i \operatorname{Res}(f(z), ia) = -\frac{\pi i e^{-ka}}{a^2}. \quad (25)$$

Thus, $\int_{-R}^{-r} f(z)dz + \int_r^R f(z)dz = -\frac{\pi i e^{-ka}}{a^2} - 0 - \left(-\frac{\pi i}{a^2}\right) = \frac{\pi i}{a^2}(1 - e^{-ka})$. Therefore, it is directly to find that $\int_{-\infty}^{+\infty} \frac{\sin(kx)}{x(x^2+a^2)} dx = \operatorname{Im}(\int_{-\infty}^{+\infty} f(z)dz) = \operatorname{Im}\left[\frac{\pi i}{a^2}(1 - e^{-ka})\right] = \frac{\pi}{a^2}(1 - e^{-ka}) \frac{\sin(kx)}{x(x^2+a^2)}$, which turns out to be an even function. Ultimately,

$$I = \int_0^{+\infty} \frac{\sin(kx)}{x(x^2+a^2)} dx = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\sin(kx)}{x(x^2+a^2)} dx = \frac{\pi}{2a^2}(1 - e^{-ka}). \quad (26)$$

3.4. Example 4

The third example considered is

$$I = P.V. \int_{-\infty}^{+\infty} \frac{\sin x}{e^x - e^{-x}} dx. \quad (27)$$

Let $f(z) = \frac{e^{iz}}{e^z - e^{-z}}$, and the singular points are $z = k\pi i$, where k is an integer. Consider the contour shown in Fig. 4, where the two singular points 0 and πi are excluded from the contour. Let C_{r1} and C_{r2} denote the two negatively oriented semi-circles. So, there's no singular points interior to this contour. Applying Eq. (13), it is arrived that

$$\lim_{r \rightarrow 0} \int_{C_{r1}} f(z)dz = -i\pi \lim_{z \rightarrow 0} zf(z) = -i\pi \frac{(ze^{iz})^{(1)}}{(e^z - e^{-z})^{(1)}} \Big|_{z=0} = -\frac{1}{2}i\pi, \quad (28)$$

And that

$$\lim_{r \rightarrow 0} \int_{C_{r2}} f(z)dz = -i\pi \lim_{z \rightarrow \pi i} (z - \pi i)f(z) = -i\pi \frac{((z - \pi i)e^{iz})^{(1)}}{(e^z - e^{-z})^{(1)}} \Big|_{z=\pi i} = \frac{\pi i e^{-\pi}}{2}. \quad (29)$$

Notice that $|e^{R+iy} - e^{-R-iy}| = \sqrt{(e^{2R} + e^{-2R} - 2) + 4\sin^2 y} \geq e^R - e^{-R}$, the integral along the two vertical paths is

$$\left| \int_B^D f(z) dz \right| = \left| \int_0^\pi \frac{ie^{i(R+iy)}}{e^{R+iy}-e^{-R-iy}} dy \right| \leq \frac{1}{e^R-e^{-R}} \int_0^\pi e^{-y} dy = \frac{1-e^{-\pi}}{e^R-e^{-R}}, \quad (30)$$

Which tends to 0 as R tends to infinity. Therefore, $\lim_{R \rightarrow \infty} \int_B^D f(z) dz = 0$ and that $\lim_{R \rightarrow \infty} \int_C^A f(z) dz = 0$.

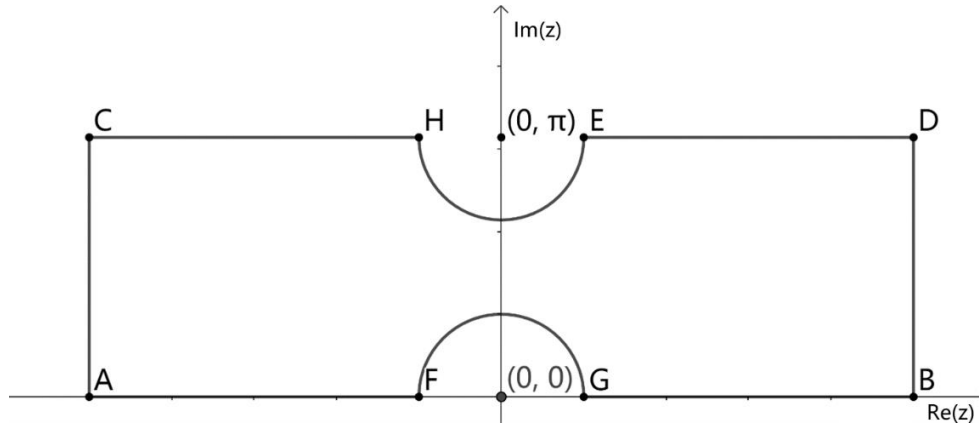


Figure 4. The contour of integral in example 4

On the other hand, consider the integral along the other paths, $\int_D^E f(z) dz + \int_H^C f(z) dz = \int_R^r \frac{e^{iz}}{e^z - e^{-z}} dz + \int_{-r}^{-R} \frac{e^{iz}}{e^z - e^{-z}} dz = e^{-\pi} \left(\int_{-R}^{-r} \frac{e^{ix}}{e^x - e^{-x}} dx + \int_r^R \frac{e^{ix}}{e^x - e^{-x}} dx \right)$. By using of Cauchy's residue theorem, it is easy to infer that the identity $\int_D^E f(z) dz + \int_{C_{r2}} f(z) dz + \int_C^A f(z) dz + \int_A^F f(z) dz + \int_{F_{r1}} f(z) dz + \int_G^B f(z) dz + \int_B^D f(z) dz = 2\pi i \cdot 0 = 0$ holds. When $R \rightarrow +\infty, r \rightarrow 0$, it becomes

$$-\frac{1}{2}i\pi + \frac{\pi ie^{-\pi}}{2} + \int_{-\infty}^{\infty} \frac{e^{ix}}{e^x - e^{-x}} dx + e^{-\pi} \int_{-\infty}^{\infty} \frac{e^{ix}}{e^x - e^{-x}} dx = 0, \quad (31)$$

And thus the integral

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{e^x - e^{-x}} dx = \frac{\frac{1}{2}i\pi + \frac{\pi ie^{-\pi}}{2}}{1 + e^{-\pi}} = \frac{1}{2}i\pi \frac{1 - e^{-\pi}}{1 + e^{-\pi}} = \frac{1}{2}i\pi \frac{\frac{\pi}{2} - e^{-\frac{\pi}{2}}}{\frac{\pi}{2} + e^{-\frac{\pi}{2}}} = \frac{1}{2}i\pi \tanh\left(\frac{\pi}{2}\right). \quad (32)$$

Therefore,

$$\int_{-\infty}^{+\infty} \frac{\sin x}{e^x - e^{-x}} dx = \text{Im} \left(\int_{-\infty}^{\infty} \frac{e^{ix}}{e^x - e^{-x}} dx \right) = \frac{1}{2}\pi \tanh\left(\frac{\pi}{2}\right). \quad (33)$$

4. Conclusion

In summary, this paper introduces the method for obtaining the residue at the singular points and then shows how to apply Cauchy's residue theorem by presenting four examples. In all the four cases, a new function with complex variables is introduced to replace the real integrand. In order to obtain the integrals, proper contours should be created accordingly. In the first example, the two singular points above the real axis are included in the contour. The contour consists of the real axis and a semi-circle with its radius tending to infinity. The integral along the semi-circle tends to zero as radius tends to infinity, so this part of integral can be eliminated. The same is true for example 2, but there are two different line segments to be considered, and the integral along the inclined line segment equals to a constant-value times the integral along the horizontal line segment. In examples 3 and example 4, the contours are generated so that the origin is excluded from the contour, since it lies on the real axis and the integrand is not defined on this point. In example 4, the point $(\pi, 0)$ is also

excluded from the simple closed contour. Also, in this case, the contour consists of several line segment and two small indented paths, but there are no sectors of a circle with its radius tends to infinity, and this is due to the feature of function, where its integral along the two vertical paths are both zero. This paper shows that Cauchy's residue theorem is a powerful tool to compute improper integral, and has several superiorities over conventional methods. Apart from the examples given in this paper, some other essential improper integral, such as the Fresnel integral and Gaussian integral can also be computed by applying Cauchy's residue theorem.

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