Proof and Application of the Mean Value Theorem

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Abstract. In calculus, mean value theorem (MVT) connects a function's derivative and its rate of change over a certain interval. This paper delves into the mathematical intricacies of the MVT and its multifaceted applications. Through rigorous proofs and illustrative examples, this study establishes the MVT's fundamental role in calculus and its relevance in understanding the behavior of functions. The paper extends its exploration to encompass related theorems, including extreme value theorem, which connects function's continuity and extrema, Intermediate Value Theorem, which states that the function value within an interval of a continuous function must be between the maximum and minimum values, local extreme value theorem, Rolle's theorem, a specific situation of the theorem, and the integral MVT, an application in integral aspect of MVT, further enriching the comprehension of these pivotal concepts. These theorems provide powerful tools for understanding the properties of continuous functions, identifying critical points, and establishing relationships between function values and their derivatives. This paper highlights the significance of proving these theorems and solving mathematical problems as applications. Through a systematic exploration of the mathematical foundations, this paper contributes to a deeper comprehension of the core principles underlying calculus and their applied theorems in different contexts.

Keywords: Extreme value theorem, Rolle's theorem, Intermediate value theorem, Mean value theorem.

1. Introduction

In calculus, mean value theorem (MVT) is a fundamental theorem that connects a function's derivative to its change of the function value over a particular closed interval of input variable. The theorem is a significant part in the basic research of calculus and has applications in various fields, including physics, engineering, and economics. In this paper, a thorough demonstration of the MVT is presented. The statement of MVT is shown below. If function f is a real continuous function on a closed regime $[a, b]$ and is differentiable on open region of (a, b), then the derivative of f at a certain point in that interval (named as point c) is thus equal to the ratio of change of $f(x)$ and change of x, i.e.,

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{1}$$

This theorem is an important study of calculus in higher mathematics, which serves as a bridge connecting functions and their derivatives. With the MVT as a tool, derivatives will be used to study the behavior of functions more easily. The Rolle's theorem is one interesting theorem that requires the boundary points of the given region to have same function values, which is a special case of the MVT. It can be used to prove the existence of equation roots. Scholars have conducted some research in this area. The method of constructing auxiliary functions using integral functions has been discussed [1]. Several notes on the Rolle's theorem have been provided [2], mainly emphasizing the conditions for applying this special MVT [3].

To give a complete and clear proof of the MVT, this paper will present several other relevant theorems, basic theorems like extreme value theorem (EVT) which connects function's continuity and extrema, and intermediate value theorem (IVT) which states that the function value within an interval of a continuous function must be between the maximum and minimum values. Some theorems needed to prove like local extreme value theorem (Local EVT) and Rolle's theorem are also involved. Also, the application on the integral MVT will also be discussed.

2. Theorems

In this section, four distinct theorems are introduced one by one, followed by a brief proof and explanation.

Theorem 2.1 (Extreme value theorem) [4]. If f is a continuous function defined in the region $[a,b]$ and x_1, x_2 belong to the aforementioned interval, then for arbitrary x in $[a,b]$, $f(x_1) \leq f(x) \leq f(x_2)$.

Theorem 2.2 (Intermediate value theorem) [5]. If f is a continuous function defined in the region $[a,b]$, for any real value R in the range $[f(a), f(b)]$, $f(x)$ reaches a value of R in $[a,b]$.

Theorem 2.3 (Local extreme value theorem) [6]. If f is defined in an open region (a,b), if $f(x)$ displays a local extremum at a position in (a,b), then the derivative of $f(x)$ at that point must be either zero or undefined.

Proof: Assume that f has a local maximum at a position in (a,b), and name that location c. According to what the derivatives are defined, one has the following relation

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}. \tag{2}$$

In what follows, it aims to show that the limit is 0 or undefined. If the limit does not exist, that is proved. Now the author assumes that the limit exists. Consider that x approaches c from left and right side. If x gets close to c from values larger than c, then $x - c$ is positive and $f(x) - f(c)$ is less than or equal zero, which means

$$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0. \tag{3}$$

For the other case, if x gets close to c from values smaller than c, then $x - c$ is negative and $f(x) - f(c) \leq 0$, which means

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0. \tag{4}$$

Combining both cases, it is obvious to infer

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = 0. \tag{5}$$

Maximum and minimum of a function can be determined using the Local EVT as an application. Given that f is a function that is defined on $[a, b]$. this theorem can find the proper x when the derivative of f is zero. Then, compare those $f(x)$'s with values of f at endpoints to find a maximum or minimum.

Theorem 2.4 (Rolle's theorem). Assume that f is continuous defined in the region $[a, b]$ and it is differentiable on the open interval. If $f(a)$ and $f(b)$ are equal, there exists a point, named point c, over the closed region $[a, b]$ so that the derivative at c is null.

Proof: Fix an arbitrary continuous function f on $[a, b]$ with the condition that the function is also differentiable on the corresponding open region. By EVT, f must have an extremum on $[a,b]$, named point c. If $c \in (a,b)$, then by Local EVT, it is found that the f' at that point is zero or does not exist. Since f is differentiable on $(a$, b), its derivative should exist, so f' at that point is zero. However, if c does not belong to (a,b), then one of a, b is maximum and the other is minimum. Since the function values at the endpoints are equal, f is constant on (a,b), the function values are all equal to zero over the open interval.

The analysis and comparison of the differences in geometric meaning between Rolle's theorem and Lagrange's theorem can also be revealed [7]. The first example is the Rolle's theorem. In a closed region $[a,b]$, there is a defined continuous function $y = f(x)$ with two equal ordinates at endpoints. Consider a tangent line intersects to the function at every point. The conclusion is that more than one point ξ on the function that makes the corresponding tangent line at ξ to the x-axis.

The other example is Lagrange's theorem. In closed intervals $[a, b]$, there is a defined continuous curve $y = f\ (x)$. Consider a tangent line intersects to the function at every points. The conclusion is that more than one-point ξ on the function that makes the tangent line at ξ parallel to the chord which is built by connecting ends of the function in that period.

Due to the condition $f(a) = f(b)$ in the first theorem, two straight lines by connecting two ends of the curve arc and the tangent at ξ all parallel to the x-axis are built. However, in Lagrange theorem, continuous function $f\ (x)$ does not need to meet the condition of equal valued endpoints, so the tangent line at ξ can only be parallel to chord of endpoints but not parallel to the x-axis. Among those two theorems, the position of ξ (may be set as ξ be the only point) is the maximum distance from the curve to the chord of endpoints. From this, it is found that the first theorem is a special case of the second theorem. The Rolle’s Theorem is useful to find the number of solutions of a function. When a function, like a high degree polynomial, is give, it’s hard to find all the solutions, but there are many strategies to estimate how many solutions it has. Use the IVT to predict the function at least has what number of solutions. Use Rolle’s Theorem to predict at most. Here, one only focuses on the second step.

Given a continuous and differentiable function f, the number of solutions of f is less than or equal to the number of solutions of $f' + 1$. To prove it, it is assumed that f is continuous and differentiable. Also, assume $f(x_1) = f(x_2) = 0$. Use Rolle’s Theorem for f on $[x_1, x_2]$, there exists a point a subjecting to $x_1 < a < x_2$ and $f\ '(a) = 0$. Then, the author can conclude that for any two solutions of f, there must be at least one solution of f' between them. After that, it is convincing to conclude that the number of zeros of f' is larger than or equal the number of zeros of $f - 1$, which can lead to the theorem above.

In higher mathematics, science plays a very important role, and the rational application of logic and the withdrawal of theoretical conclusions is a very charming and fulfilling thing. Derivative, as an important tool for studying the properties and mapping relationships of functions, is also the foundation for studying functions. However, starting from the existing concept theory of derivative alone, it seems that it cannot fully reflect the role of the tools it represents. It needs to be established on the basis of some basic theorems in differential calculus, such as the generalized two theorems discussed above and Cauchy's theorem, to achieve research work. And these theorems are collectively form the differential mean value theorem. Among these mean value theorems, the Rolle's theorem serves as a fundamental theorem, generalizing other important theorems. As a form of generalization of the Rolle's theorem, the Lagrange's theorem is a critical and widely used theorem in differential calculus. That theorem serves as a bridge between functions and their derivatives, serving as a means of communication, it is also a scientific tool to use the locality of derivatives to study the integrity of function mapping relationships. Therefore, exploring new forms of generalization of the Rolle's theorem is more important in higher education, especially in higher mathematics [8].

3. Mean Value Theorem and Its Application

3.1. Mean Value Theorem

Theorem 3.1 (MVT) [9]. If a real continuous function f on a region $[a, b]$, and it is differentiable on (a, b), the derivative of f at a certain point in that interval (named as point *c*) is then equal to the ratio of change of $f\ (x)$ and change of x

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{6}$$

Proof: The proof begins by fixing a function $f(x)$ that can be differentiated on (a, b) and continuous on $[a, b]$, see Fig. 1 for illustration.

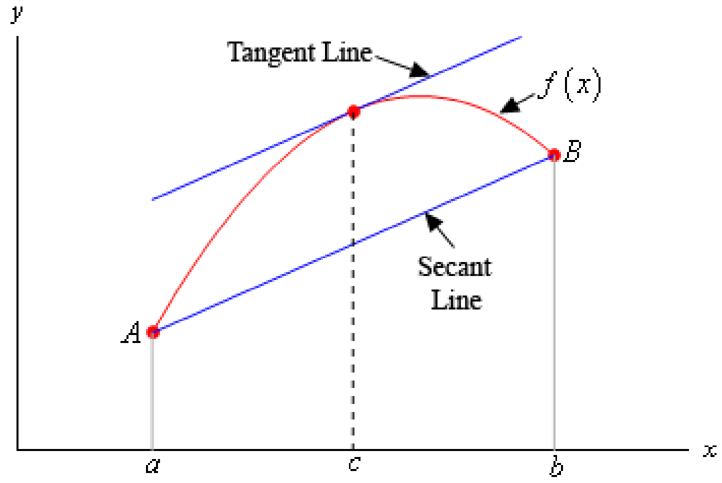

Fig. 1: The schematic diagram for proving MVT [10].

The red curve in Fig. 1 is $f(x)$, and values a, b are also shown in the figure. The blue line (secant line) is termed as $g(x)$, which intersects with $f(x)$ at points A, B on the graph. Define a constant m which is the slope of $g(x)$, then one has

$$m = \frac{f(b) - f(a)}{b - a}. \tag{7}$$

For x belongs to an open region (a, b), build another function $h(x)$, where $h(x)$ is the difference of $f(x)$ and $g(x)$, and $g(x)$ in the graph can be shown as $f(a) + m(x - a)$. Therefore, $h(x) = f(x) - g(x) = f(x) - f(a) - m(x - a)$. Hence, h is continuous on $[a, b]$ and differentiable on (a, b). Also, $h(a)$ and $h(b)$ are equal to zero. Hence, according to Rolle's theorem, there exists a point c in the interval (a, b) such that the derivative of that point is zero. Notice that $h'(x) = f'(x) - m$, $f'(c) = m = \frac{f(b)-f(a)}{b-a}$.

This theorem can be used to check whether a function constant or not by getting derivatives. Let f defined on $[a, b]$. If f is continuous on $[a, b]$, and for all x in that open interval, the derivative of f at x is zero, then f is constant on $[a, b]$. To prove it, let $\forall x_1, x_2 \in [a, b]$, $f(x_1) = f(x_2)$. Fix arbitrary $x_1, x_2 \in [a, b]$, assume $x_1 < x_2$. To use MVT for f on $[x_1, x_2]$, first verify the hypothesis: f is continuous on $[a, b]$, so f is continuous on $[x_1, x_2] \subseteq [a, b]$. f is differentiable on (a, b), so f is continuous on $(x1, x2) \subseteq (a, b)$. Thus, by MVT, a value c that is bounded by x_1 and x_2 in an open interval should exist, with

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}. \tag{8}$$

However, the derivative of f at c is zero, so $f(x_2) = f(x_1)$.

An obvious application is the L'Hopitals rule. Suppose that f and g are continuous and differentiable at a, that $g'(a) \neq 0, f(a) = g(a) = 0$. Under these assumptions, one finds that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{\lim_{x \to a} \frac{f(x) - f(a)}{x - a}}{\lim_{x \to a} \frac{g(x) - g(a)}{x - a}} = \frac{f'(a)}{g'(a)}. \tag{9}$$

Under the assumption, f' and g' are continuous at a, and it is a similar sitiution: $\lim_{x \to a} \frac{f'(x)}{g'(x)}$. Note that this is just a particular case of the result, but it is one of the most used cases, and it is widely applicable, for example, it also holds of analytic functions on an open domain in the plane, a case definitely not covered by the usual (real valued) argument.

3.2. Integral MVT

Theorem 3.2 (Integral MVT). If f is a continuous curve on the period [a, b], and it is differentiable on (a, b), then the value of function f at a certain position in that interval (named as point c) can be expressed as [11]

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\ dx. \tag{10}$$

The identity in Eq. (10) can be rewriten in a other way for easier understanding

$$\int_a^b f(x)\ dx = f(c)(b-a). \tag{11}$$

Proof (Method 1): The EVT states that $f(x)$ has minimum and maximum on that closed interval, since it is continuous [11]. Therefore, let's set m as the minimum value of f and M as maximum. Then, for any x in that interval, have $m \le f(x) \le M$. Consequently, in light of the definition of integral,

$$m(b-a) \le \int_a^b f(x)\ dx \le M(b-a). \tag{12}$$

Divide all parts by (b – a), it is found that $m \le \frac{1}{b-a}\int_a^b f(x)\ dx \le M$. Then, $\frac{1}{b-a}\int_a^b f(x)\ dx$ is a value between minimum and maximum values of f. Also, since the function is continuous and defined on $[a, b]$, the Eq. (10) holds according to the IVT.

Proof (Method 2): For all x in [a, b], set two new functions f_1, f_2 as $f_1(x) = (x-a)\int_a^b f(x)\ dx$, and $f_2(x) = (b-a)\int_a^x f(t)\ dt$. Then, define f_3 as $f_3(x) = f_1(x) - f_2(x)$, and note that $f_3(a)$ and $f_3(b)$ are equal to zero. Due to Rolle's Theorem, have $f_3'(c)$ equal to zero, where c is a specific point in that interval [12]. Since

$$f_3{}'(x) = \int_a^b f(x)\ dx - (b-a)f(x) \tag{13}$$

Then $\int_a^b f(x)\ dx = f(c)(b-a)$.

This theorem has many applications. On the one hand, it is helpful to solve this limit [13]

$$L = \lim_{n\to\infty} \mathrm{n}^2 \int_{((\mathrm{n}+1)!)^{-1/(n+1)}}^{((\mathrm{n})!)^{-1/n}} \mathrm{h(nx)\ dx}. \tag{14}$$

Therefore, the goal is that if the function f is a continuous curve on (a, b) and $e \in (a, b)$, then

$$L = \lim_{n\to\infty} n^2 \int_{((n+1)!)^{-1/(n+1)}}^{((n)!)^{-1/n}} f(nx)\ dx = ef(e). \tag{15}$$

Let $b_n = n(n!)^{-1/n}, a_n = n\big((n+1)!\big)^{-1/(n+1)}$, and by the integral MVT it is arrived that

$$L = \lim_{n\to\infty} n^2 \int_{((n+1)!)^{-1/(n+1)}}^{((n)!)^{-1/n}} f(nx)\ dx = n\int_{a_n}^{b_n} f(t)\ dt = n(b_n - a_n)f(t_n) \tag{16}$$

is true for a certain $t_n \in (a_n, b_n)$. Then, by stirling formula one has

$$b_n = ne^{-\frac{\ln(n!)}{n}} = e - \frac{e\ln n}{2n} - \frac{e\ln\sqrt{2\pi}}{n} + O\left(\frac{\ln^2 n}{n^2}\right), \tag{17}$$

and thus $b_n - a_n = b_n - \frac{nb_{n+1}}{n+1} = \frac{e}{n} + O\left(\frac{\ln n}{n^2}\right) = e$. This means that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} t_n = e$. By the assumption that f is continuous at e, then

$$\lim_{n\to\infty} n(b_n - a_n) f(t_n) = ef(e). \tag{18}$$

Here, function h from the original equation is a real continuous function on (0, +∞), so the limit is $e\, h(e)$.

On the other hand, this theorem can be used to prove the Taylor's theorem, also known as the Taylor's formula with the error term [14]. It is a result that approximates a function using its derivatives at a specific point and an error term. The last term is used to be presented as the higher-order derivatives and a remainder. The remainder is often written in the Cauchy form using the MVT. Starting with the Taylor series expansion for a arbitrary function named f around $x = 0$, it is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots. \tag{19}$$

Here, $f^{(n)}(0)$ denotes the n-th derivative of function f, which is around $x = 0$. Note that this derivation assumes certain smoothness and differentiability conditions on f and its derivatives. The integral representation of Taylor series expansion is give by $f(x) = f(0) + \int_0^x f'(x-t)\, dt$. Differentiate the whole equation with repect to x, it is found that $f'(x) = f'(0) - \int_0^x f''(x-t)\, dt$.

Now, the MVT for definite integrals can be used. According to the theorem, for a continuous function $g(t)$ on the close interval $[a, b]$, for some point named c in $[a, b]$, it is found that $\int_a^b g(t)\, dt = g(c)(b-a)$ [15]. Applying this to the intergral above, it is concluded that

$$\int_0^x f''(x-t)\, dt = f''(c)(x-0) = xf''(c). \tag{20}$$

Substituting this back into the equation for the first derivative, it is $f'(x) = f'(0) - xf''(c)$. Finally, integrating this equation with reprect to x from 0 to x, it is arrived that

$$f(x) = f(0) + \int_0^x \left(f'(0) - xf''(c)\right) dx = f(0) + xf'(0) - \frac{x^2}{2}f''(c). \tag{21}$$

Here, the term $\frac{x^2}{2}f''(c)$ is the error term in the Taylor expansion, often denoted by $R_2(x)$, which explains the discrepancy between the approximate value of the function and the actual value. This shows the Cauchy form for the remainder. The Cauchy form tells that the remainder is proportional to x^2 and a function's second derivative at some point c between 0 and x.

4. Conclusion

In conclusion, the MVT stands as a fundamental pillar in the realm of calculus, serving as a bridge between the derivative and the change of a function value over the change of variable. Through the course of this paper, the author delved into the intricate mathematical details of the MVT. The journey through the MVT has not been undertaken in isolation; it has led researchers to closely examine related theorems such as the Rolle’s theorem. These interconnected theorems, each with its unique insights, provide powerful tools for discerning the properties of continuous functions, identifying critical points, and establishing crucial relationships between function values and their derivatives.

Furthermore, the exploration extended to the MVT for integrals, shedding light on its significance in the realm of integrals. This variant has further emphasized the elegance and universality of the MVT, showcasing its adaptability to diverse mathematical contexts. By navigating through these foundational concepts and theorems, the author have deepened people’s understanding of calculus. In essence, this paper has offered a comprehensive exploration of the MVT and its extensions, providing a solid foundation for appreciating the intricacies of calculus and its invaluable role in shaping understanding of functions, rates of change, and so on. As people conclude this study, they acknowledge the enduring importance of these theorems and their applications, fostering a continued fascination with the beauty and utility of mathematical principles.

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