The Existence and Stability of Limit Cycle of Some Special Types of 2D Autonomous Dynamic Systems

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Abstract. This article is focusing on the problem about limit cycles of 2D autonomous dynamics, which is a part of the famous Hilbert's 16th problem. There are mainly three results on this paper. The first two results discuss the cases with sum of squares of the variables. Through polar substitution and the mathematics analysis on the explicit limit cycle obtained by Cramer's Rule, the first result reveals the uniqueness and stability of the limit cycle of a specific form for 2D autonomous dynamic system. For the second part of the results, by changing the coefficients of the system of the first result, it is found that the stability of limit cycle remains unchanged on the phase portrait. The third part gives a special polynomial with terms of general odd degrees for Liénard equation, which can be rewritten as a case of 2D autonomous dynamic system. Without explicit expression of the limit cycle, the Poincaré–Bendixson method and the Liénard theorem help to give the conclusion on both uniqueness and stability of this system.

Keywords: Limit cycle; dynamics; two-dimension autonomous system; stability.

1. Introduction

Consider a 2D autonomous dynamical system:

$$\begin{cases} \dot{x_1} = P(x_1, x_2) \\ \dot{x_2} = Q(x_1, x_2) \end{cases}, \tag{1}$$

where $(x_1, x_2) \in \mathbb{R}^2$, $P(x_1, x_2)$ and $Q(x_1, x_2)$ are functions on the phase portrait, which have continuous partial derivatives in order one. When P and Q are linear functions, there are clear results on all behaviors on the phase portrait of the system [1]. However, it is believed that for most of the general cases, there are no explicit solutions for system (1). As one part of the 16th problem that the famous German mathematician David Hilbert issued in 1900, even for the cases with simple forms of P and Q, i.e., quadratic polynomials of x_1 and x_2, there are still many crucially unsolved problems for the behaviors of (1), mainly including its stability of equilibrium points and the numbers of limit cycles [2]. Generally, the strong nonlinearity of system (1) will lead to different kinds of bifurcation and even chaos, which can bring much difficulty to give a detailed mathematical analysis [3]. For some specific forms of P and Q, methods on studying some important and typical models may give many inspirations.

The system of predator–prey equations for species:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(x_1, x_2)\boldsymbol{x}, \tag{2}$$

Where $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\boldsymbol{f}(x_1, x_2) = \begin{bmatrix} M(x_1, x_2) & 0 \\ 0 & N(x_1, x_2) \end{bmatrix}$, $M(x_1, x_2)$ and $N(x_1, x_2)$ are continuous function of non-negative variables x_1 and x_2, with continuous partial derivatives. One of the famous examples is Lotka-Volterra model with,

$$\boldsymbol{f}(x_1, x_2) = \begin{bmatrix} a - bx_2 & 0 \\ 0 & -c + dx_1 \end{bmatrix}, \tag{3}$$

Where a, b, c and d are positive constants.

By linearization of the system, there are some information of the equilibrium points. When $x_1 = x_2 = 0$, a saddle node and the nullclines are obtained on its corresponding phase portrait. However, the method of linearization for the equilibrium points is not enough to determine an important behavior of its solutions: whether they could spiral toward the equilibrium point or outward to a limit cycle or else lie on closed orbits. To make this determination, a subtle skill is to construct a Lyapunov function $L(x_1, x_2)$ as a linear combination of $x_1, x_2, \log x_1$ and $\log x_2$. For further, it is shown that every solution of Lotka-Volterra model is a closed orbit [1].

Another typical example is the Van der Pol equation, it is a non-linear ODE with order. As a special case of a more general differential equation called Liénard equation,

$$\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} + f(x_1)\frac{\mathrm{d}x_1}{\mathrm{d}t} + \mathrm{g}(x_1) = 0, \tag{4}$$

Via integral substitution, the equation can be regarded as a 2D autonomous dynamical system,

$$\begin{cases} \dot{x_1} = x_2 - F(x_1) \\ \dot{x_2} = -\mathrm{g}(x_1) \end{cases}, \tag{5}$$

Where $\frac{\mathrm{d}F(x_1)}{\mathrm{d}x_1} = f(x_1)$ and $x_2 = \dot{x_1} + F(x_1)$. Once $f(x)$ and $\mathrm{g}(x)$ locally satisfy a series conditions including Lipschitz continuity, under a suitable construction of Poincaré map and index, it is shown that system (5) will have a unique periodic solution. Details in [1, 4].

The following will introduce the cases of P and Q are quadratic polynomials with single variable with respect to x_1 and x_2 respectively. Through a classification of different situations of the discriminants of P and Q, a criterion of monotonicity and special stability of solutions of (1) with explicit solutions of the Cauchy problem was shown. About the existence of isolated singular points and qualitative behavior of the general solutions, it shows two calculation conclusions. Specially, some applications on the relative mathematical models of oil fields and waterflooding are shown with regulating and monitoring the related process [2].

For the problems on looking for WR_0 realization, specially focusing on the cases with P and Q are polynomial of any orders with single variables, under a related complex-balanced condition, an algorithm on computer give us a result on its existence and uniqueness [5].

About the problems of locations and numbers of limit cycles of (1), the main idea is constructing a special map applied on Poincaré-Miranda Theorem (PMT), and there is a counterexample in the form of a degree 6 polynomial for Kouchnirenko conjecture. The relating method can be applied to plenty of examples like a special type of counterexamples to Markus-Yamabe conjecture, i.e., the existence of three limit cycles for a 2D vector field [6].

The above examples showed us some methods of studying the behaviors and existence of limit cycles. For the coefficient matrix in a rotational form, or with terms like “$x_1^2 + x_2^2$”, employing a substitution with polar coordinates is a usual skill. For more complicated cases, using a suitable Lyapunov function seems more subtle, since there is still not a general hint for its construction. Other cases can be seen with many different methods recently. For example, Hopf’s method combined with homoclinic bifurcation theory and the method of inverse integral factor [7]. Another old method to make limit cycles is by perturbing a system which has a center. The main idea is that the limit cycles bifurcate the perturbed system from some of the periodic trajectories of the center of the unperturbed system [8].

Inspired by the above examples and methods, our main purpose is to construct a criterion of existence and stability of limit cycles, especially for those 2D polynomial of x_1 and x_2 in degree 3 with more general coefficients and $x_1^2 + x_2^2$. We will also try to focus on a type of system (1) with odd degree polynomials that can be written into a Liénard equation.

2. Methods

This section will give some basic theorems with examples on analyzing problems about limit cycle that will be used in the following part about the results. Section 2.1 is going to give the mathematical analysis on limit cycle for the cases that an explicit solution can be found in polar coordinates. Although most of the cases of (1) have no explicit solutions, the related methods are still basically important for developing new ideas. Through an example on constructing a bounded region for the trajectories, instead of analytically giving an explicit expression, section 2.2 will show some applications of the famous Poincaré–Bendixson method to determine some possible behaviors of (1). In section 2.3, Liénard equation and the related theorem about uniqueness and stability of limit cycle will show important inspiration on a case of Q being a polynomial with odd degrees.

2.1. Mathematical Analysis under an Explicit Solution

The mathematical analysis with explicit solutions shown here is modified by an important example from [1]. For system (1), with real constant a, b, c and d, consider

$$\begin{cases} P(x_1, x_2) = x_1 + x_2 - x_1(ax_1^2 + bx_2^2) \\ Q(x_1, x_2) = -x_1 + x_2 - x_2(cx_1^2 + dx_2^2) \end{cases}. \tag{6}$$

Let $a = b = c = d = 1$, by polar coordinates substitution $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$, system (1) with such condition (6) can be rewritten into

$$\begin{cases} \dot{r} = r(1 - r^2) \\ \dot{\theta} = -1 \end{cases}. \tag{7}$$

For an initial condition that $\theta = 0$ when $t = t_0$, where $t_0 \in \mathbb{R}$, we mainly focus on one of the explicit solutions $r = 1$ and $\theta = -t + t_0$ for $t \geq t_0$. This is an isolated periodic solution (thus it is a limit cycle) with periodic 2π. By analyzing the vector directions as time goes to infinity of other trajectories on the phase portrait, the stability and uniqueness of the limit cycle in a compact region of ring-shaped can be shown.

2.2. Poincaré–Bendixson Method

For most cases of (1), explicit solutions cannot be analytically found globally even in a local region. Inspired by the analysis of section 2.1, it is natural to consider building a closed region without equilibrium points, better in the shape of circular ring, to "trap" a limit cycle. Once a trajectory enters the region, it seems like to be "trapped" as it cannot intersect with itself again and thus, it will never leave the region. Named by Poincaré–Bendixson, the following theorem 1 is the core tool, and it is necessary in the following analysis. Normally, there are some equivalent statements for the theorem, and here will give a useful one of that for convenience. The proof can be found in [1]. Another theorem 2 called Bendixson negative criterion is a suitably good method to deny the existence of limit cycles of system (1). By subtly using the Green Theorem, skills of its proof are very concise and beautiful [10].

Theorem 1. Suppose D is a closed and bounded ring-shaped region on the 2D plane excluding any equilibrium points of (1), one of the trajectories of (1) passes through a point on D. Then this trajectory is either periodic or spirals towards a periodic solution.

Theorem 2. Suppose D is a simply connected region on the 2D plane where $\frac{\partial P}{\partial x_1} + \frac{\partial Q}{\partial x_2}$ is not identically null and sign-unchanged, then system (1) has no periodic solutions on region D.

Two examples will briefly give applications on the above two theorems. Slightly change the form of (6) under the similar skills from [10], let $a = c = 2$ and $b = d = 1$, by polar coordinates substitution $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$, system (1) with the corresponding condition of (6) can be rewritten as:

$$\begin{cases} \dot{r} = r[1 - r^2(1 + \cos^2\theta)] \\ \dot{\theta} = -1 \end{cases}. \tag{8}$$

Different from the cases of section 2.1, and to the best can be found, it is challenging to give an explicit and simple solution for system (8). Note that $1 \leq 1 + \cos^2\theta \leq 2$ for all $\theta \in \mathbb{R}$. From the first equation of system (9), it is clear that:

$$r(1 - 2r^2) \leq \dot{r} \leq r(1 - r^2). \tag{9}$$

Therefore, consider a ring-shaped region:

$$D = \{(r, \theta) | \ \epsilon_1 \leq r \leq \epsilon_2\}, \tag{10}$$

Where $0 < \epsilon_1 \leq 1/\sqrt{2}$ and $\epsilon_2 \geq 1$. Easy to know the origin is the only equilibrium point of system (8), region D excludes the origin. Under theorem 1, a clear mathematical analysis of all trajectories trapped in D can be shown to illustrate the existence of limit cycles of system (8).

Let$a = -1$, $c = 3$ and $b = d = 1$, system (1) with condition (6) can be changed into:

$$\begin{cases} \dot{x_1} = x_1 + x_2 - x_1(-x_1^2 + x_2^2) \\ \dot{x}_2 = -x_1 + x_2 - x_2(3x_1^2 + x_2^2) \end{cases}, \tag{11}$$

It is clear that $\frac{\partial P}{\partial x_1} + \frac{\partial Q}{\partial x_2}$ remains positive in a small enough circle centered at the origin. Under theorem 2, there is no periodic solution for system (11).

2.3. Liénard Theorem

Since the stability of limit cycles corresponds to whether a physical phenomenon about periodic oscillation can be obtained, there is a practical significance to explore more methods on the problems about limit cycles of some specific forms of (1). French physicist Liénard gave a theorem to illustrate a special type of second order ODE will have unique stable limit cycle.

Theorem 3. In systems (4) and (5), suppose: (i) $f(x)$ and $g(x)$ are continuous for all $x \in \mathbb{R}$, and $g(x)$ locally satisfy Lipschitz condition, (ii) $f(x)$ is even function with $f(0) < 0$; $g(x)$ is odd function and $xg(x) > 0$ when $x \neq 0$,(iii) $\lim_{x\to\pm\infty} F(x) = \pm\infty$, $F(x)$ has unique positive zero at a, and $F(x)$ is nondecreasing for $x \geq a$. Then system (5) has a unique stable limit cycle.

The proof of theorem 3 is a complicated process, but a Chinese mathematician Zhang gave a stronger statement with detailed proof of theorem 3 [11]. Once system (1) satisfies the conditions of theorem 3, the trajectories on the phase plane are symmetrical around the origin, which is also the only equilibrium point.

For system (4), consider:

$$\begin{cases} f(x_1) = \lambda(x_1^2 - 1) \\ g(x_1) = x_1 \end{cases}, \tag{12}$$

Where λ is a real constant, under which system (4) becomes the Van der Pol equation. Easy to check Van der Pol equation satisfies all conditions of theorem 3, it has a unique stable limit cycle. This example shows an analyzing method with theorem 3. In the next section, results on some higher dimension of (1) will be based on this idea.

3. Results and Discussion

This section will give results on situations with different P and Q of system (1) and similar examples in [12]. Polar substitution will be widely used in the cases with $x_1^2 + x_2^2$ for both analytic and implicit solutions. For solving the 2×2 system of $\dot{r}$ and $\dot{\theta}$, Cramer's Rule is an important method for the simplification of the explicit form of $\dot{r}$ and $\dot{\theta}$. By adjusting different values of a, b, c and d for system (6), the first two parts in the following will show same stabilities of the limit cycles.

3.1. $P(x_1, x_2)$ and $Q(x_1, x_2)$ in System (1) with $x_1^2 + x_2^2$

Consider special coefficients on system (6), for the case $b = a$ and $d = c$, system (6) can be written as:

$$\begin{cases} P(x_1, x_2) = x_1 + x_2 - ax_1(x_1^2 + x_2^2) \\ Q(x_1, x_2) = -x_1 + x_2 - cx_2(x_1^2 + x_2^2) \end{cases}. \tag{13}$$

By polar coordinates substitution $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$, system (1) with (13) becomes a 2×2 linear system with respect to $\dot{r}$ and $\dot{\theta}$,

$$\begin{cases} (\cos\theta)\dot{r} - x_2\dot{\theta} = x_1 + x_2 - ax_1r^2 \\ (\sin\theta)\dot{r} + x_1\dot{\theta} = -x_1 + x_2 - cx_2r^2 \end{cases}. \tag{14}$$

Without losing the generality, assume $r > 0$, system (14) can be solved by Cramer's Rule.

$$\begin{cases} \dot{r} = \dfrac{\begin{vmatrix} x_1+x_2-ax_1r^2 & -x_2 \\ -x_1+x_2-cx_2r^2 & x_1 \end{vmatrix}}{\begin{vmatrix} \cos\theta & -x_2 \\ \sin\theta & x_1 \end{vmatrix}} = r[1 - r^2(a\cos^2\theta + c\sin^2\theta)] \\ \dot{\theta} = \dfrac{\begin{vmatrix} \cos\theta & x_1+x_2-ax_1r^2 \\ \sin\theta & -x_1+x_2-cx_2r^2 \end{vmatrix}}{\begin{vmatrix} \cos\theta & -x_2 \\ \sin\theta & x_1 \end{vmatrix}} = -1 + r^2(a-c)\sin\theta\cos\theta \end{cases}. \tag{15}$$

Specially, when $a = c$, system (14) can be furtherly written as:

$$\begin{cases} \dot{r} = r(1 - ar^2) \\ \dot{\theta} = -1 \end{cases}. \tag{16}$$

For $a \le 0$, $r = 0$ is a special solution and the unique equilibrium point as well as the origin. For $a > 0$, there are two special solutions:

$$\begin{cases} r = 0 \\ \dot{\theta} = -1 \end{cases}, \tag{17}$$

And

$$\begin{cases} r = 1/\sqrt{a} \\ \dot{\theta} = -1 \end{cases}. \tag{18}$$

The second solution is a limit cycle in the shape of a circle, denoted as circle C, centered at the origin. With an explicit solution, the stability of this limit cycle can be given via the discussion of the trajectories on the phase portrait. By polar substitution, for an initial condition, assume an orbit passes through point $(\tilde{r}, \tilde{\theta})$. When $\tilde{r} < 1/\sqrt{a}$, $\dot{r} = r(1 - ar^2) > 0$. These orbits seem like being "trapped" into circle C with increasing radius. $\dot{\theta} = -1 < 0$ implies a clockwise rotational spiral, which means all trapped orbits are approaching circle C. Similarly, when $\tilde{r} > 1/\sqrt{a}$, $\dot{r} = r(1 - ar^2) < 0$, orbits lie outside circle C, with decreasing radius and clockwise rotating situation. All these orbits also tend to circle C. As time t tends to infinity, once both inside and outside orbits are approaching to the limit cycle, this limit cycle is called stable.

3.2. $P(x_1, x_2)$ and $Q(x_1, x_2)$ with Special Constructions in Degree 3

For a case of $P(x_1, x_2)$ and $Q(x_1, x_2)$ with a special construction when $b = c = 0$ and $d = a$ in system (6), we can solve the expressions of $\dot{r}$ and $\dot{\theta}$ by Cramer's Rule like 3.1. However, unlike 3.1, both $\dot{r}$ and $\dot{\theta}$ can only be expressed in terms of r and θ instead of analytics or explicit solutions. Consider:

$$\begin{cases} P(x_1, x_2) = x_1 + x_2 - ax_1^3 \\ Q(x_1, x_2) = -x_1 + x_2 - ax_2^3 \end{cases}, \tag{19}$$

Where a is a positive constant. Note that:

$$\frac{\partial P}{\partial x_1}+\frac{\partial Q}{\partial x_2}=2-3a(x_1^2+x_2^2), \tag{20}$$

In circular region $x_1^2+x_2^2\le 2/3a$, $\frac{\partial P}{\partial x_1}+\frac{\partial Q}{\partial x_2}>0$. By Theorem 2, the system (1) with condition (19) has no periodic solution. For a further discussion in the region out of $x_1^2+x_2^2\le 2/3a$, we can try to express $\dot r$ and $\dot\theta$ by Cramer's Rule like 3.1. By polar coordinates substitution $x_1=r\cos\theta$ and $x_2=r\sin\theta$, system (1) with (19) becomes a 2×2 linear system with respect to $\dot r$ and $\dot\theta$,

$$\begin{cases}(\cos\theta)\dot r-(r\sin\theta)\dot\theta=x_1+x_2-ax_1^3\\(\sin\theta)\dot r+(r\cos\theta)\dot\theta=-x_1+x_2-ax_2^3\end{cases}. \tag{21}$$

Express $\dot r$ and $\dot\theta$ by Cramer's Rule:

$$\begin{cases}\dot r=\dfrac{\begin{vmatrix}x_1+x_2-ax_1^3 & -x_2\\ -x_1+x_2-ax_2^3 & x_1\end{vmatrix}}{\begin{vmatrix}\cos\theta & -x_2\\ \sin\theta & x_1\end{vmatrix}}=r[1-ar^2(\sin^4\theta+\cos^4\theta)]\\ \dot\theta=\dfrac{\begin{vmatrix}\cos\theta & x_1+x_2-ax_1^3\\ \sin\theta & -x_1+x_2-ax_2^3\end{vmatrix}}{\begin{vmatrix}\cos\theta & -x_2\\ \sin\theta & x_1\end{vmatrix}}=-1+ar^2\sin\theta\cos\theta(\cos^2\theta-\sin^2\theta)\end{cases}, \tag{22}$$

The same as 3.1, for $\dot r=0$, $r=0$ is an equilibrium point. Another special solution is:

$$r=\frac{1}{\sqrt{a(\sin^4\theta+\cos^4\theta)}}, \tag{23}$$

It is easy to have $\sin^4\theta+\cos^4\theta=1-\frac{1}{2}\sin^2 2\theta\in\left[\frac{1}{2},1\right]$, then $1/\sqrt{a}\le r\le\sqrt{2}/\sqrt{a}$. Note that:

$$\sin\theta\cos\theta(\cos^2\theta-\sin^2\theta)=\frac{1}{4}\sin 4\theta\in\left[-\frac{1}{4},\frac{1}{4}\right], \tag{24}$$

and$ar^2\in[1,2]$, then:

$$\dot\theta=-1+\frac{1}{4}ar^2\sin 4\theta<0. \tag{25}$$

It is clear that $1/\sqrt{a}\le r\le\sqrt{2}/\sqrt{a}$ is a closed and bounded ring-shaped region on the 2D plane excluding equilibrium points. When the trajectory passes through this region, by Theorem 1, there must be a limit cycle for this system. Although special solution (23) is not explicit, instead, it is expressed in terms of θ, we can still see it is trapped in ring-shaped region $1/\sqrt{a}\le r\le\sqrt{2}/\sqrt{a}$. By Theorem 1, there must be a limit cycle. Furthermore, for its stability, consider the inner boundary $r=1/\sqrt{a}$ of the region, $\dot r=\frac{1}{2}r\sin^2 2\theta>0$, $\dot\theta=-1+\frac{1}{4}\sin 4\theta<0$. The trajectory moves spirally clockwise from $r=1/\sqrt{a}$ into the region. Meanwhile, in the outer boundary $r=\sqrt{2}/\sqrt{a}$, $\dot r=r(-1+\sin^2 2\theta)<0$, $\dot\theta=-1+\frac{1}{2}\sin 4\theta<0$, similarly, the trajectory moves from the outer boundary into the region. These two phenomena imply that the limit cycle is still stable.

3.3. $P(x_1,x_2)=x_2$ and $Q(x_1,x_2)$ is an Odd Polynomial in System (1)

Inspired by an example in [10], with more general coefficient for a special type of odd polynomial and a term of degree 3 of x_1 and x_2, consider system (1) with the following condition:

$$\begin{cases}P(x_1,x_2)=x_2\\Q(x_1,x_2)=-p(x_1)+\beta x_2-\gamma x_1^2x_2\end{cases}, \tag{26}$$

Where all α_i $(i\in\mathbb{N}^+)$, β and γ are positive real constants, for any positive integer n, define:

$$p(x_1)=\sum_{i=1}^{n}\alpha_i x_1^{2i-1}, \tag{27}$$

System (1) can be written as:

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = -p(x_1) + \beta x_2 - \gamma x_1^2 x_2 \end{cases}. \tag{28}$$

Note that:

$$\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} = \dot{x_2} = -p(x_1) + \beta \dot{x_1} - \gamma x_1^2 \dot{x_1},$$

Then system (28) can be regarded as a Liénard equation ritten as:

$$\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} + (\gamma x_1^2 - \beta)\frac{\mathrm{d}x_1}{\mathrm{d}t} + p(x_1) = 0. \tag{29}$$

Let$f(x_1) = \gamma x_1^2 - \beta$, easy to check $f(x_1)$ is an even function and continuous in $\mathbb{R}$, $f(0) < 0$. Note that $p(x_1)$ is a polynomial whose terms all have odd degrees, which implies $p(x_1)$ is an odd function. Easy to observe that:

$$x_1 p(x_1) = x_1 \sum_{i=1}^{n} \alpha_i x_1^{2i-1} = \sum_{i=1}^{n} \alpha_i x_1^{2i} > 0. \tag{30}$$

Denote:

$$F(x_1) = \int_0^{x_1} f(t)\mathrm{d}t = \frac{1}{3}\gamma x_1^3 - \beta x_1, \tag{31}$$

for$F(x_1) = 0$, the roots are 0, $-\sqrt{\frac{3\beta}{\gamma}}$ and $\sqrt{\frac{3\beta}{\gamma}}$, among them, there is exactly one positive zero. When $x_1 \geq \sqrt{\frac{3\beta}{\gamma}}$, $F'(x_1) = \gamma x_1^2 - \beta \geq \gamma\left(\sqrt{\frac{3\beta}{\gamma}}\right)^2 - \beta = 2\beta > 0$, which implies $F(x_1)$ is strictly increasing on the interval $\left(\sqrt{\frac{3\beta}{\gamma}}, +\infty\right)$. When $x_1 \longrightarrow +\infty$, clear that $F(x_1) \longrightarrow +\infty$. The above discussion shows that system (28) satisfies all conditions of Theorem 3, then system (28) has a unique stable limit cycle.

3.4. Disccusion

The exploration for the number and stability of limit cycle for system (1) with different types of P and Q is still a challenging problem. The above results showed the existence and stability of limit cycles in some cases with specific polynomials mainly including terms of $x_1^2 + x_2^2$. Therefore, polar substitution can be widely used for its explicit solutions. However, for other cases of system (6) with more general conditions, such as different values of a, b, c and d, it is natural to ask whether there will be similar conclusions for those cases? Or will there be chaos when a, b, c and d continuously change? Like section 3.3, can we give more types of $f(x_1)$ and $\mathrm{g}(x_1)$ for system (4) or (5) to obtain similar properties of their limits cycles? It is believed that any new methods on more general forms of P and Q will give crucial inspiration for the Hilbert's 16th problem.

4. Conclusion

This paper considers the problem about properties, including existence, number and stability, of the limit cycle of 2D autonomous dynamic system. The author first gave a summary for the importance of this problem. As one part of the famous Hilbert's 16$^{\text{th}}$ problem, this problem is still attracting much attention from mathematicians and researchers all over the world. Some 2D autonomous dynamic system can be written as a non-linear ordinary differential equation with order 2 called Liénard equation. The related theorems of Liénard equation give crucial inspiration for this paper's results.

The main idea of this paper is based on the generalization of some concrete forms of the 2D autonomous dynamic system. The first two results are given by listing some specific forms with different coefficients of (6). It is found that the special solutions of system (1) with different $P(x_1, x_2)$ and $Q(x_1, x_2)$ can be conditionally expressed depending on a, b, c and d of system (6). If there is an explicit solution, a detailed mathematical analysis on its limit cycle is given. Although sections 3.1 and 3.2 show different solvability on system (6) with different conditions, they give same stability for their limit cycles. Finally, section 3.3 gives a specific polynomial with terms of general odd degrees for Liénard equation, which can be rewritten as a case of system (1). As a beautiful tool of the determination for both uniqueness and stability of limit cycles, theorem 3 immediately helps to give the result of case from section 3.3.

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