Chaotic Phenomenal and One-dimensional Logistic Map

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Abstract. Introducing the basic properties of logistic map. The paper includes the proofing of the stability of fixed points in different domain of r and determining the property of period doubling of Logistic map, describing Chaos phenomenon in logistic map by showing the stability of fixed points in different intervals of r, researching 3-period limit cycle in the Logistic Map bifurcation diagram, showing a special characteristic of the logistic map on the region on the bifurcation diagram that exist 3-period limit cycles, collecting the different value of r at one, two, four, eight, sixteen and thirty-two pitchfork points from the Logistic Map Bifurcation diagram to calculate Feigenbaum constant and proving the Feigenbaum can also be used to determine the pitchfork bifurcation speed of logistic map when r is between zero to negative two. Showing the basic properties of the Mandelbrot Set. Showing the result of computer simulation in real part to the Mandelbrot Set to determine the relationship between the Logistic Map bifurcation diagram and Mandelbrot Set.

Keywords: Logistic map; chaos theory; Feigenbaum constants; Bifurcation diagram; Mandelbrot set.

1. Introduction

Many scientists believe that there exists a mathematical formula to correspond the variation of weather, population, planet orbits and temperature. The population growth or decay of an island also took place by a general formula. If you have taken a Calculus course before, you must be familiar with the logistic equation, an “S” curve used to describe population growth. As the logistic equation moves a step forward, biologist Rovert popularized the famous ideology in polynomial mapping called logistic map in a 1976 paper [1]. A discrete-time demographic model similar to a Logistic map by rewritten logistic equation was presented mathematically by Pierre François Verhulst [2]. A logistic map is a linearized model that usually is used in predicting population growth. The population varies with the different growing rates of “r”.

In most cases, the behavior of the population variation graph will irregularly oscillate between two points, they population maximum and population minimum respectively. Since the horizontal axis represents generation n, in which n is the positive integer starting from zero, ending in infinity generations. The vertical axis of the logistic map is the population. For all n_i, where the subscript represents the zero and each positive integer follow the order from zero to infinity. One n corresponds with an iterated population. Thus, Logistic map is a map in a discrete system, it visualizes the population at each moment. To further understand and explore the logistic map, the property of unpredictability of Chao Theory emerged in the logistic map as the rate of population growth - represented by variable r and iteration n goes to an infinite term, the outcome of the population iteration will become dramatically different since people pick different number of r [3, 4]. This phenomenon shows one of the properties of Chao Theory which is the sensitivity to the initial value. This behavior which generated by logistic parabola is also called period chaos [5].

Furthermore, Feigenbaum Constants, an amazing ratio between each pitchfork points of the Logistic Map bifurcation diagram are introduced in the following paper [4]. Additionally, by research about the graph of the Mandelbrot Set in the three-dimensional space, an astonishing discovery is that the graph of the intersection plane in the location where the left half part of Mandelbrot set symmetry to the right half part in three-dimensional space is highly similar to the Bifurcation Diagram of Logistic Map. Base on this discovery, this paper aims to determine the relationship between the Logistic map and the Mandelbrot set.

2. Logistic Map

2.1. Properties of Logistic Map

2.1.1 Behavior of logistic map

There are some properties of a logistic map we need to know first. The basic logistic map form is shown below:

$$x_{n+1} = rx_n(1 - x_n). \tag{1}$$

$x_n \in (0\,,1)$, $r \in (-2\,,4)$, $n = \{0,1,2,3,\dots\}$, x_{n+1} is the result of iteration. Assume $f^a(x) = x_{n+1}$, $a = \{0,1,2,3,4,\dots\}$; $f^2(x) = r\big(f(x)\big)\big(1 - f(x)\big)$; $f^3(x) = r\big(f^2(x)\big)\big(1 - f^2(x)\big)$; $f^4(x) = r\big(f^4(x)\big)\big(1 - f^4(x)\big)$. Remember that a $f^a(x)$ is not the derivative of $f(x)$.

2.1.2 Calculation of the fixed point of the Logistic Map

Stable fixed point exists when $r \in (0\,,3\,)$. The equation of fixed point of Logistic Map$f(x)$ is given by:

$$rx(1 - x) = x. \tag{2}$$

$$\left(x - 1 + \frac{1}{r}\right) = 0. \tag{3}$$

$$x = 0 \; and \; x = 1 - \frac{1}{r}. \tag{4}$$

It shows that $f(x)$ has a fixed point at 0 when $r \in (0\,,1\,)$, Otherwise, $f(x)$ exist a fixed point when $r > 1$. Next, we are going to determine the stability of $f(x)$ by taking its first derivation. $f'(x) = r(1 - 2x)$ The fixed-point x $= 0$ is stable and attracting when $0 \leq f'(x) \leq 1$, Thus the value of r should also fall into $0 \leq r \leq 1$. So when r is greater than 1, fined point $x = 0$ becomes unstable and repelling, it is explained by the definition of stability of fixed point, when $r > 1$, $x = 1 - \frac{1}{r}$ will be greater than 0, in which making fixed being unstable. Furthermore, when we graph $x_{n+1} = rx_n(1 - x_n)$, and $y = x$. There is only one intersection point between two functions when $0 \leq r \leq 1$. "A 2-cycle exists if and only if there are two points p and q such that $f(p) = q$ and $f(q) = p$. Equivalently, such a p must satisfy $f^2(p) = p$ where $f(x) = x(1 - x)$. Hence, p is a fixed point of the second iterate map $f^2(u_{n+1}) = u_{n+1}$. Since f(x) is a quadratic map, $f^2(x)$ is a quartic polynomial." [6].

By plugging the fixed point $x = 0$ and $x = 1 - \frac{1}{r}$ in the first derivation equation of the Logistic map, we get:

$$f'(0) = r \text{ and } f'\left(1 - \frac{1}{r}\right) = 2 - \frac{1}{r}. \tag{5}$$

We see that, $x = 0$ is stable when $1 > r \geq 0$, it is unstable when $1 < r$. For the $x = 1 - \frac{1}{r}$, it is stable when $3 > r > 1$, and it is unstable when $r > 3$, two cycles appear in the second iteration of the Logistic map, in which $f^2(x)$:

$$r(rx(1 - x)\big(1 - rx(1 - x)\big) = 0. \tag{6}$$

$$-r^3 * x^4 + 2r^3x^3 + (-r^3 - r^2)x^2 + r^2x = 0. \tag{7}$$

Then we divide the trivial solution $x_1 = 0$ and $x_2 = 1 - \frac{1}{r}$, we will get:

$$x^2 - \left(\frac{r+1}{r}\right)x + \frac{r+1}{r^2} = 0. \tag{8}$$

Then we can use discriminant to figure out p, and q separately,

$$p, q = \frac{r+1 \pm \sqrt{(r-3)(r+1)}}{2r}. \tag{9}$$

$$\left(\frac{(r+1)}{r}\right)^2 \geq \frac{4(r+1)}{r^2}. \quad (10)$$

Therefore, p_1, q_1 only exist when $r \geq 3$. This also indicate that 2 cycle exist for $r \geq 3$. But if we want to find an attractor, we need to find:

$$f'(p_1)f'(p_2) = (r - 2rp_1)(r - 2rp_2). \quad (11)$$

$$r^2(1 - 2\left(\frac{r+1}{r}\right) + 4\left(\frac{r+1}{r}\right). \quad (12)$$

$$-r^2 + 2r + 4. \quad (13)$$

When we put $r = 3$ in $-r^2 + 2r + 4$, we got 1. So, for $r = 3$, it decreases monotonically. And it reaches to 0 when $r = 1 + \sqrt{5}$, and reaches to -1, when $r = 1 + \sqrt{r}$. Thus, the 2-period cycle is attractive when attractive $3 < r < 1 + \sqrt{6}$ and becomes a repellent for $r > 1 + \sqrt{6}$.
As we take $r \in (-1, 1)$, x_{n+1} will converge to 0 as n approach ∞, no matter what value of x_n you pick. In this case, we call 0 is the stable cycle in $r \in (-1, 1)$, the bifurcation diagram of logistic map is shown below (Figure 1 and 2).

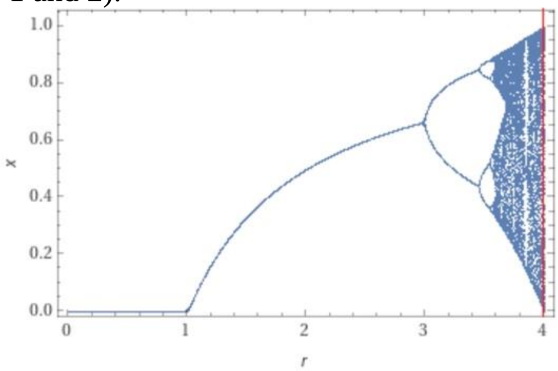

Fig. 1 Bifurcation Diagram r = 4

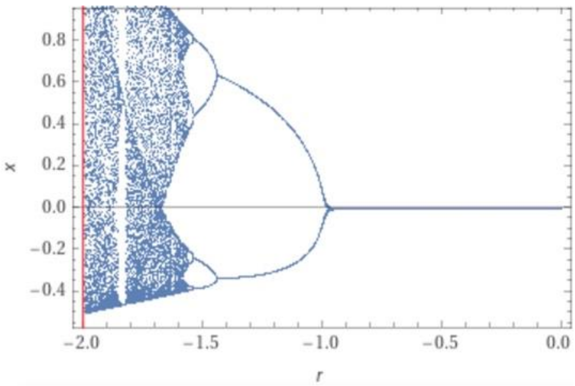

Fig. 2 Bifurcation Diagram r = -2

Take $r \in (1, 3)$, x_{n+1} will converge to a number. The trajectory of the converged value is an increasing curve. Randomly take $x_0 = 0.45$, and From $r = 3$ to $r \approx 1 + \sqrt{6}$, the values of x_{n+1} fellows the Table 1 below:

Table 1. Iteration results

n	n_{n+1}	n	n_{n+1}
0	0.45000	7	0.84587
1	0.85375	8	0.44973
2	0.43071	9	0.85366
3	0.84581	10	0.43093
4	0.44986	11	0.84592
5	0.85370	12	0.44961
6	0.43082	13	0.85361

x_{n+1} is oscillating between 0.45 and 0.85. This behavior also called two periods limit cycle. From $r_2 \approx 1 + \sqrt{6} \approx 3.449$ to $r_3 \approx 3.544$, x_{n+1} has a four periods limit cycle. From $r_4 \approx 3.544$ to $r_5 \approx 3.564$, x_{n+1} has an eight periods limit cycle, then 16 periods limit cycle, then 32 limits cycle, and so on This behavior is called the periodic doubling. After $r \approx 3.56995, x_{n+1}$ emerge chaotic behavior [5]. Taking $r \in (3.829\,, 3.841)$, then zoom in the bifurcation diagram, we can see there exist a place that has a 3 periods limit cycle or onset of period three.

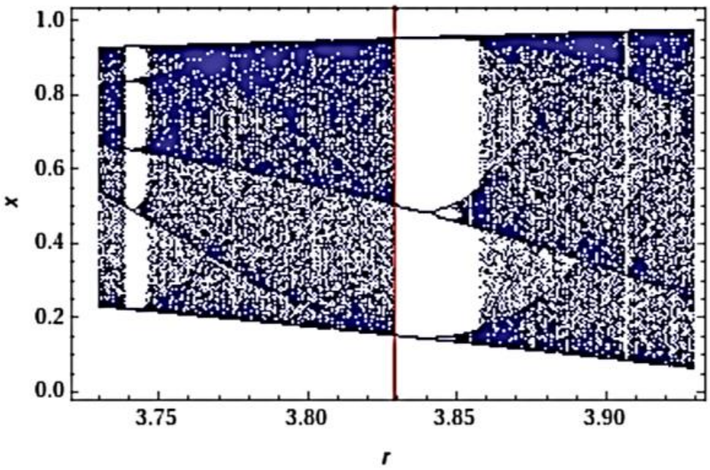

Fig. 3 Bifurcation Diagram $r = 3.82$

When we continually zoom in the space of 3-periods limit cycle, and look at the pitchfork bifurcation, there is a small size of the bifurcation diagram that is similar with the initial diagram, as we keep zoom in, we will find another three-period limit cycle interval. This phenomenon will reappear again and again [7-9].

2.2. Determination of δ between Each Pitchfork Point in $r \in (-2, 0)$ & $r \in (0, 4)$

If we take the distance L1 between start point of limit cycle 1, which is $r_1 \approx 3$ by the start point of limit cycle 2, $r_2 \approx 1 + \sqrt{6} \approx 3.4495$ is equal to $L1 = 3.449 - 3 = 0.449$. $L2$ is the distance between the start point of limit cycle 2 to limit cycle 4, $r_3 - r_2$, which is: $L2 = 3.544 - 3.449 = 0.095$. Let $L1 / L2$, we get $\delta 1 = 0.449 / 0.095 = 4.72632$. The distance between the start point of limit cycle 16, $r_4 \approx 3.5644$ and the start point of limit cycle 8, $r_3 \approx 3.544$ is L3 = 3.5644 – 3.545 = 0.0194. The distance between the start points of limit cycle 32, $r_5 \approx 3.568$. The start point of limit cycle 16 is $r_5 - r_4 = L_4 = 3.568 - 3.5644 = 0.0036$, thus $\delta 2 = L4/L3 = 5.389$.

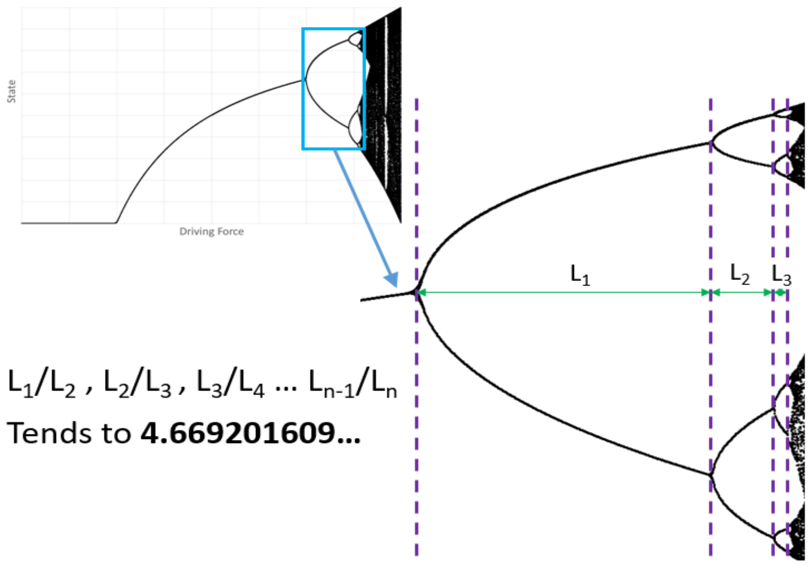

Fig. 4 Bifurcation diagram [10]

In general, we take

$$\lim_{n\to\infty} \frac{r_{n-1}-r_{n-2}}{r_n-r_{n-1}}. \quad (14)$$

We will get a number close to 4.7, which is average 4.669201 by the computer simulation. This is called Feigenbaum constants, where r_n is used for describing the value of r at nth term [4]. This interesting phenomenon indicates that the speed of population bifurcates is the approximately same when $r \in (-1.5681, -1)$ and $r \in (1,3.5699)$.

2.3. Proofing Feigenbaum Constants Fit for r $\in (-2, -1)$

What if we chose the value of r $\in (-2, 02)$? We will get a small size of Logistic map. Next, we are proofing Feigenbaum Constants, which doesn't apply when r $\in (-1, 0)$.

Proof. Here, we use a_n denotes as the start point of bifurcation diagram parameter r when r $\in (-2, 0)$. We start at two - period cycle point of r, because $r \in (-1, 0)$ has a fixed point 0, which is meaningless.

The 2- limit period cycle, $a_1 \approx$ -1; 4 limit period cycle $a_2 \approx -1.4494$, $L1 = -1 - (-1.4494) = 0.4494$. 8 limit period cycle, $a_3 \approx -1.541, L2 = -1.4494 - (-1.541) = 0.0916$. $\delta_1 = \lim_{n\to\infty} \frac{r_{n-1}-r_{n-2}}{r_n-r_{n-1}} = L1/L2 = 4.906$. The 16-limit period cycle, $a_{16} \approx -1.562$, so $L3 = -1.541 - (-1.562) = 0.021$. Thus $\delta_2 = \lim_{n\to\infty} \frac{r_{n-1}-r_{n-2}}{r_n-r_{n-1}} = L2 \frac{-L30.0916}{0.021} = 4.361$. $\Delta = (\delta_2 + \delta_1)/2 = 4.6335$. Then we take average value of ratio $\delta_2$and δ_1, we got 4.6335. Therefore, the common ratio of Logistic map bifurcation is 4.6335. Those data are recorded by hand. Thus, there exists error. However, when we compare 4.6335 and 4.669201, they are closed [4].

3. Definition of Mandelbrot Set

Mandelbrot set is the set that all points are in the complex plane. It forms by self-iteration. Complex plane defined as a composition of a vertical imaginary axis and a horizontal real axis. The general formula for the Mandelbrot can be written as $z_{n+1} = z_n^2 + C$. Where C is a complex number, in which $C \in \mathbb{C}$ and $C = a + bi$, where $a, b \in \mathbb{R}$, and $z_1 = C$ [7]. The point C only exist in the interval where $|z_n| \leq 2$, if the absolute value of z_n is greater than 2, the iteration will diverge to infinity in its end behavior [8].

The iteration is shown below. $z_2 = z_1^2 + C$; $z_3 = z_2^2 + C; z_4 = z_3^2 + C; z_5 = z_4^2 + C$. This is Mandelbrot set in the three-dimensional space.

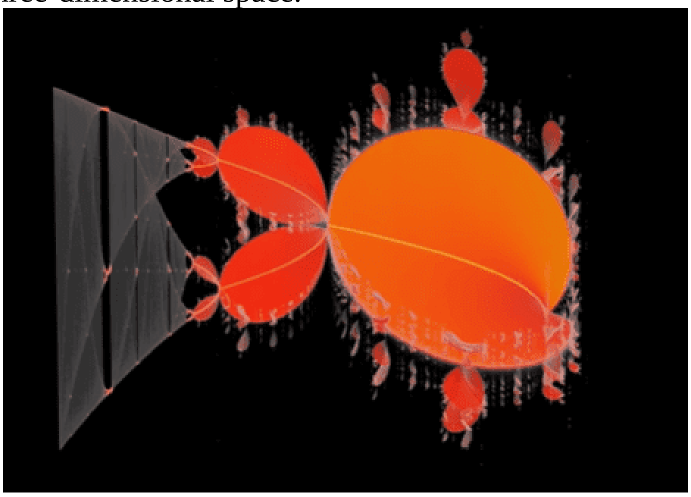

Fig. 5 Mandelbrot Set [8]

Following the yellow bifurcation diagram, it seems almost the same as the Logistic Map. The picture above is the quadratic map which is composed of the real part and complex part of the

Mandelbrot Set. If we merely look at the real part of the $z_{n+1} = z_n^2 + C$, and graph it in the 2-dimensional space or just look at the vertical intersection of the quadratic map, we will see a graph which is highly similar as the Logistic Map Bifurcation diagram.

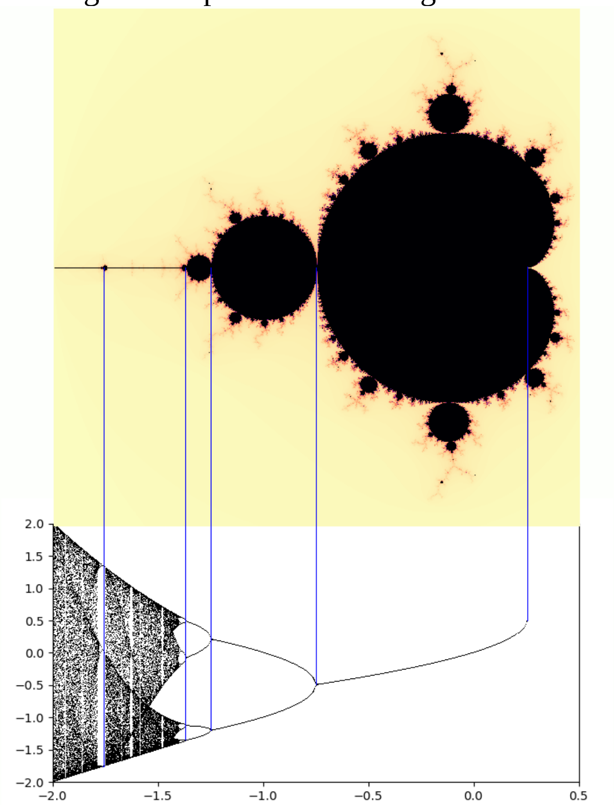

Fig. 6 Mandelbrot Set and Its Bifurcation Diagram [9]

The horizontal axis represents z value, and y-axis represent z_{n+1}. All the points that fall into the black region will converge to certain number, and emerge limit cycles. Beyond the convergent region, all the points will diverge to infinity. From about -0.75< x < 0.25, Mandelbrot has one limit cycle and converge to a value, it is similar with the Logistic Map from $r \in (1,3)$. From about -1.25 < x <-0.75, it is 2 limit cycles. In the Logistic Map, it corresponds to $r \in (1,1+\sqrt{6})$. From about -1.75 < x <-1.25, four limit cycles emerged, in the Logistic Map, it is $r \in (1+\sqrt{6}, 3.544)$, and so on. 3 periods limit cycle not only appear in the Logistic Map, but also in the Mandelbrot Map. In the Logistic Map, three periods limit cycle is approximately located at $r \approx 3.829$. Mandelbrot Set has a three periods limit cycle at x = 1.75 [8].

4. Conclusion

This paper is used for introducing a Logistic Map to readers. This paper shows how to utilize the property of the first derivative of the logistic map equation to prove the stability of the fixed points, in which 1 limit cycle is stable when $1 > r \geq 0$, and 2 limit cycle is stable when $3 < r < 1 + \sqrt{6}$, the value of r where 4-period limit cycles, 8-period limit cycles, 16-period cycles, and 32-period limit cycles appeared in the logistic map bifurcation diagram are obtained by the approximation from online graph Wolfram. The paper shows the proof that Feigenbaum Constant can be also used to determine the ratio of each Pitchfork bifurcation point in the region of the bifurcation diagram which $r \in (-2,1)$ is approximately 4.633. Those data are obtained by using the observation in the Wolfram online figure, thus, the data collection has certain errors. This value will get closer to 4.669201 if this paper uses Python or other programs to obtain the data. Furthermore, an interesting fact is that if we only look at the real part of the Mandelbrot Set, corresponding it to the Bifurcation diagram, people can see the real part of the Mandelbrot set is highly similar to the bifurcation diagram of the logistic map.

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