Cauchy Residue Theorem and K-residue Theorem

Xiaomin Jiang *

No.1 High School, Anhui, 230000, China

* Corresponding author: jiangmin@tianhelaw.cn

Abstract. Due to negative numbers do not have square roots, scientists introduced complex numbers, which are more abstract compared with real numbers. Residue Theorem has a very significant status in complex analysis – it can be used to simplify difficult integrals. In this article, Cauchy's Residue Theorem is first introduced with definition and proof. Then the author shows two examples of using the theorem to solve problems, which can help readers to better explain the theorem. After that, K-residue Theorem is introduced, which is a special kind of Residue Theorem, with a special function called K-function. Finally, the theorem is used to solve a problem. These problems are difficult to solve without Residue Theorem. Through the analysis, it is easy to find the importance of Residue Theorem: it is very useful, not only in math but also in many other fields like physics, biology, and chemistry. Moreover, it still has more potential for people to develop.

Keywords: Residue Theorem, K-residue Theorem, Integral.

1. Introduction

Complex analysis is a branch of complex functions. It's mainly about the properties and features of complex functions. The basic concept of complex analysis is complex numbers, which enlarge the range of numbers. Complex analysis enables scientists to work out the questions that cannot be solved in real variable functions. Complex analysis involves analytic function, holomorphic function, harmonic function, conformal map, etc. Within the complex analysis, Residue Theorem is a pretty useful tool used to compute the path integral of an analytic function along a closed curve, and can also be used to compute the integral of a real function. It is a generalization of Cauchy's Integral Theorem and Cauchy's Integral Formula. Residue Theorem is very important. It has a lot of applications. For example, the use of the Residue Theorem was reevaluated by Dai and Zhang in the real integration of a kind of function that decays fast [1]. Szenes calculated Residue Theorem for rational trigonometric sums and Verlinde's formula [2]. Gupta, Talwar and Verma computed exponential excitation response of electric network circuits through Residue Theorem [3].

In this paper, Residue Theorem is the main research target. In 1825, Cauchy figured out the problems about complex integrals, based on an analogy with the case of the problem of calculating real integrals, and defined residue. In 1971, Gohberg and Sigal obtained the operator generalization of the theorem on the logarithmic residue for meromorphic operator functions [4]. In 1995, Harvey and Lawson deduced a wide variety of geometric Residue Theorems [5]. In 2000, with the help of Residue Theorem, Beck derived an expression for the number of lattice points in a dilated n-dimensional tetrahedron, when there are vertices at lattice points on each coordinate axis and the origin [6]. In 2018, Alsumiri, Li, Jiang, et al. proposed a Residue Theorem with the foundation of a soft sliding mode control strategy for a PMSG (permanent magnet synchronous generator) based on a wind power generation system. They maximized energy conversion, so the system could work better [7]. In 2021, Liu and Shao summarized some common ways of computing real integrals using Residue Theorem with two specific real integrations [8].

In the following parts, Cauchy's Residue Theorem is introduced in section 2, and two examples are shown; K-residue Theorem is introduced in section 3, and one example is shown then.

2. Residue Theorem

2.1. Cauchy's Residue Theorem

To calculate the anti-derivative of complex numbers, those common methods that are used to calculate the integral of real numbers are sometimes not useable. As a result, some particular methods are developed. For example, Cauchy's Residue Theorem.

Theorem 2.1 ([9]) Let $f(z)$ be a close path, contour C, and every point inside the close path is differentiable except $z_1, z_2, \dots, z_p$ $(p \in Z)$, then

$$\int_C f(z)dz = 2\pi i \sum_{p=1}^{n} Res_{z=z_p} \mathrm{F}(z), \tag{1}$$

Where $Res_{z=z_p} f(z)$ are the residues of points $z_1, z_2, \dots, z_p$.

In order to get the residues of the points, partial fraction needs to be used. $f(z)$ can be written into $f(z) = \frac{g(z)}{h(z)}$, and since the denominator cannot be zero, $f(x)$ is not differentiable when $h(z) = 0$. As a result, $f(z)$ can be written into $f(z) = \frac{g(z)}{(z-z_1)(z-z_2)\dots(z-z_p)} = \frac{A}{z-z_1} + \frac{B}{z-z_2} + \cdots$, where $z_1, z_2, \dots, z_p$ are the roots of $h(z) = 0$, and $A, B, \dots$ are the residues of points $z_1, z_2, \dots, z_p$.

Proof: Let the points z_p $(k \in Z)$ be the centers of counterclockwise circles C_k, which is inside contour C. C_k are too small that none of them will intersect with each other. A closed region formed between C_k and C is differentiable at everywhere. According to Cauchy-Goursat theorem,

$$\int_C f(z)dz - \sum_{k=1}^{n} \int_{C_k} f(z)dz = 0. \tag{2}$$

So,

$$\int_{C_k} f(z)dz = 2\pi i Res_{z=z_p} \mathrm{f}(z) \qquad (k \in Z). \tag{3}$$

2.2. Examples and Computation

Example 2.1 Let $f(z) = \frac{az^3+bz^2+cz+d}{z^4-1}$, with $a = 6, b = i+1, c = 16, d = 1-i$. Evaluate the integrals $\int f(z)dz$, where

$$\gamma(t) = i + \frac{e^{it}}{2} \qquad 0 \le t \le 2\pi \tag{4}$$

$$\gamma(t) = \frac{i-1}{2} + \sqrt{2}e^{it} \qquad 0 \le t \le 2\pi \tag{5}$$

$$\gamma(t) = 1 + 5e^{it} \qquad 0 \le t \le 2\pi \tag{6}$$

Proof: The non-differentiable points of the function are the roots of $z^4 - 1 = 0$.

$$z^4 - 1 = (z-1)(z-i)(z+1)(z+i) = 0, \tag{7}$$

$$z_1 = 1, z_2 = i, z_3 = -1, z_4 = -i. \tag{8}$$

Let $\frac{6z^3+(i+1)z^2+16z+1-i}{z^4-1} = \frac{A}{z-1} + \frac{B}{z-i} + \frac{C}{z+1} + \frac{D}{z+i}$.

Subsequently,

$$\frac{A}{z-1}+\frac{B}{z-i}+\frac{C}{z+1}+\frac{D}{z+i} = \frac{A(z^3+z^2+z+1)+B(z^3+iz^2-z-i)+C(z^3-z^2+z-1)+D(z^3-iz^2-z+i)}{z^4-1}. \tag{9}$$

It is easy to have

$$\begin{cases} A+B+C+D=6 \\ A+Bi-C-Di=i+1 \\ A-B+C-D=16 \\ A-Bi-C+Di=1-i \end{cases} \qquad \begin{cases} A=6 \\ B=-2 \\ C=5 \\ D=-3 \end{cases}. \tag{10}$$

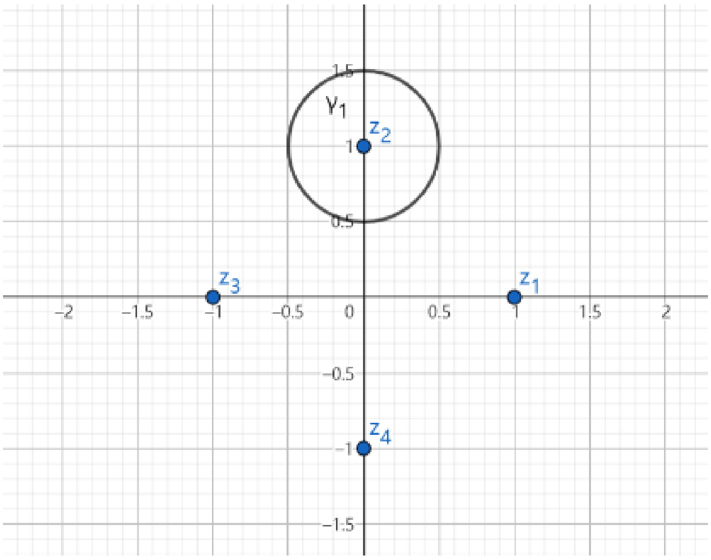

Figure 1. The contour of γ_1

For situation 1, where $\gamma(t)=i+\frac{e^{it}}{2}$ $(0\le t\le 2\pi)$, the graph of γ is shown in Fig 1. Only z_2 is included on the contour, so

$$\int f(z)dz = 2\pi i(B) = 2\pi i(-2) = -4\pi i. \tag{11}$$

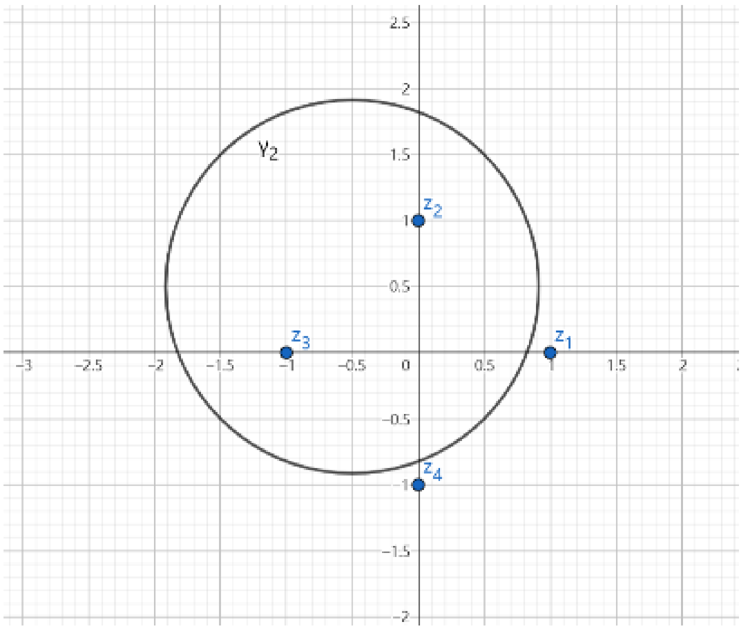

Figure 2. The contour of γ_2

For situation 2, where $\gamma(t) = \frac{i-1}{2} + \sqrt{2}e^{it}$ $(0 \le t \le 2\pi)$, the graph of γ is shown in Fig 2. Only z_2, z_3 are included on the contour, so

$$\int f(z)dz = 2\pi i(B + C) = 2\pi i(-2 + 5) = 6\pi i. \tag{12}$$

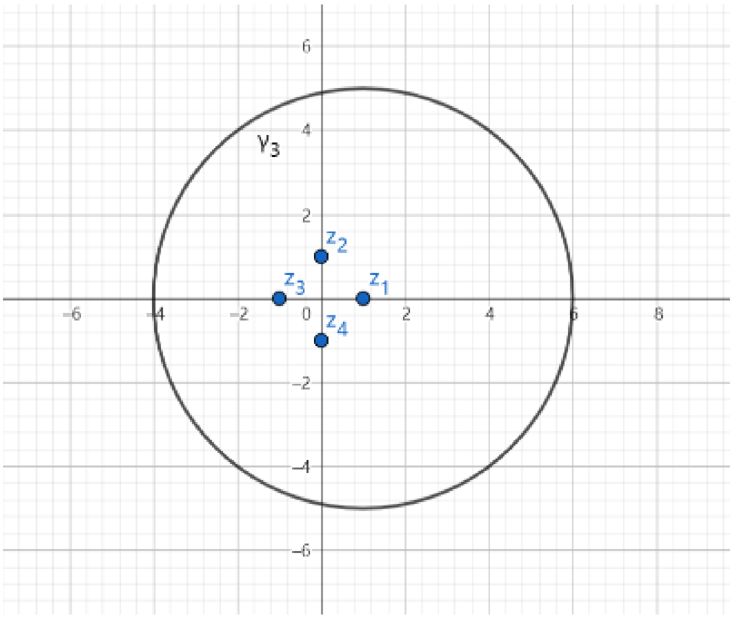

Figure 3. The contour of γ_3

For situation 3, where $\gamma(t) = 1 + 5e^{it}$ $(0 \le t \le 2\pi)$, the graph of γ is shown in Fig 3. z_1, z_2, z_3, z_4 are included on the contour, so

$$\int f(z)dz = 2\pi i(A + B + C + D) = 2\pi i(6 - 2 + 5 - 3) = 12\pi i. \tag{13}$$

Example 2.2 Calculate $\int_0^\infty \frac{\cos mx}{z^2+a^2} dx$.

Proof: Let $z^2 + a^2 = 0$ get the singularities of the function $F(z)e^{imz} = \frac{1}{z^2+a^2}e^{imz}$, which are $+ai$ and $-ai$. The residue of $\frac{e^{imz}}{z^2+a^2}$ at $+ai$ is

$$\lim_{z\to ai}\left[(z - ai)\frac{e^{imz}}{z^2+a^2}\right] = \lim_{z\to ai}\left[\frac{e^{imz}}{z+ai}\right] = \frac{e^{-ma}}{2ai}, \tag{14}$$

So

$$\int_0^\infty \frac{\cos ms}{z^2+a^2} dx = \pi i \frac{e^{-ma}}{2ai} = \frac{\pi}{2a}e^{-ma}. \tag{15}$$

3. K-Residue Theorem

3.1. The Definition of K-Residei Theorem

Definition 3.1 ([10]) The K-complex numbers of complex numbers $z = x + iy$ are $z(k) = x + iky(K \in R, k \ne 0)$. If $f(z)$ is defined at z_0, $f_{(k)}(z_0) = \lim_{z\to z_0}\frac{f(z)-f(z_0)}{z(k)-z_0(k)}$ is the K-derivative of $f(z)$ at z_0, and every point in the domain is K-differentiable.

If every points of $f(z)$ in D have their K-derivatives, $f(z)$ is analytic in D; if $f(z)$ is K-differentiable at some area when $z = z_0$, $f(z)$ is K-analytic at z_0. All the functions that are K-differentiable in D can be written as $F(D(k))$.

Let $a \in C$, $f(z)$ is K-analytic in $B(a,R)(k) - \{a\}: 0 < |(z-a)(k)| < R$, then

$$\frac{1}{2\pi i}\int_{\gamma} f(z)dz(k), \quad \text{where } \gamma: |(z-a)(k)| = \rho(\forall\rho: 0 < \rho < R) \tag{16}$$

Is the K-residue at a.

Definition 3.2 Let $f(z)$ be K-analytic in contour C, except points $a_1, a_2, \dots a_n$, and K-continuous on $\overline{D} = D + C$ except points $a_1, a_2, \dots a_n$. Then

$$\int_C f(z)dz(k) = 2\pi i \sum_{j=1}^{n} K - Res\left[f(z), a_j\right]. \tag{17}$$

3.2. Examples and Computation

Example 3.1 Calculate $I = \int_0^{\pi} \frac{d\theta}{a+cos\theta} (a > 1)$.

Proof: Let $z(k) = e^{i\theta}(0 \le \theta \le 2\pi)$, then $cos\theta = \frac{(z(k)+z^{-1}(k))}{2}$, and contour $C: |z(k)| = 1, d\theta = [i \cdot z(k)]^{-1}dz(k)$. Since the function is even, it is easy to get

$$I = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a+cos\theta} = \frac{1}{2} \int_C \frac{dz(k)}{iz(k)[a+\frac{z(k)+z^{-1}(k)}{2}]} = -i\int_C \frac{dz(k)}{z^2(k)+2az(k)+1}. \tag{18}$$

Because $z^2(k) + 2az(k) + 1 = 0$ has 2 different roots, which are $\alpha = -a + \sqrt{a^2-1}, \beta = -a - \sqrt{a^2-1}$, while $\alpha \cdot \beta = 1, |\alpha| < |\beta|$, so $|\alpha| < 1, |\beta| > 1$. As a result,

$$I = -i2\pi i \cdot K - Res[f(z), \alpha] = \left.\frac{2\pi}{2z(k)+2a}\right|_{z(k)=\alpha} = \frac{2\pi}{-2a+2\sqrt{a^2-1}+2a} = \frac{\pi}{\sqrt{a^2-1}}. \tag{19}$$

4. Conclusion

Residue Theorem is the main topic of this paper, which mainly talks about two parts of Residue Theorem: the first one is Cauchy's Residue Theorem, while the second one is K-residue Theorem. Cauchy's Residue Theorem is the most general form of Residue Theorem, which can be used to calculate the integral of complex numbers, and there are two examples to further explain it. K-residue Theorem is defined with a special function K, which is quite different from the common one. There is also an example. This paper shows people how to use residue theorem to solve questions, and shows the importance of it. The residue theorem is a tool used to calculate contour integrals, and can also borrow the properties of contour integrals to calculate the integrals of real functions. Through the residue theorem, complex waveform and circuit problems can be solved, as well as solutions can be found by calculating certain complex calculus problems, which is of great significance for research in various fields.

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