A generalization of the Taylor's theorem

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Abstract. In mathematics, Taylor's theorem is a formula that uses the information of a function to describe its proximity to a point. If the function is smooth enough, the conductive values can be used to construct a multiform to approximate the the function in the neighboring area of that point, which can even be extended to the convergence radius of the scale. Taylor’s theorem also gives the deviation between this multiform and the actual function value. The essay includes the proof and application of Taylor’s theorem and the Taylor’s theorem in multivariate functions. Special functions are set with unknown coefficients to deduce the Taylor’s theorem. Specific questions are solved by using the Taylor’s theorem as some examples. Taylor's theorem is frequently applied in real-world settings to solve problems. For instance, in engineering, it can be used to build a variety of numerical simulations and optimization algorithms. In physics, it can be used to analyze data from physical experiments. In finance, it can be used to manage risk and set prices for financial transactions.

Keywords: Taylor's theorem, Peano's form of remainder, Lagrange's form of remainder, multivariate functions.

1. Introduction

Calculus has two primary subfields: differential calculus and integral calculus. While the latter estimates the quantities and areas for curves, the former deals with instantaneous rates. Both fields employ the core concepts of infinite sequences and infinite theorem convergent to a well-defined limit, which are connected by the calculus fundamental theorem. Taylor’s theorem is an indispensable part of the calculus. A polynomial of degree n, often known as the nth-order Taylor polynomial, is used by Taylor's theorem in calculus to approximate an n-times differentiable function that revolves around a given point.

This paper mainly focuses on the Taylor's theorem, including the development and the current research status. Taylor's theorem is named after the mathematician Taylor, who first articulated a version of it in 1715. In 1986, Jiang et al. took the theoretically simple output of the two-dimensional casting plate to the point at any moment, the relationship between its plate speed and its angle of bending, as well as the direction of the plate speed, which is theoretical proof of Taylor's formula. At the same time expands the scope of the formula, it applies not only to the extreme cases of throwing [1]. In 1990, Zvonimir et al. believed that the simplest way to address the first issue is to locate a proof of Taylor's theorem that generates the Taylor polynomial to eliminate the need for an independent motive for the polynomial [2]. In 2003, Taylor's formula with Peano's residue, although the rest is only a qualitative description of the residue and cannot be calculated quantitatively, plays an important role in the theoretical proof of the gradient and the estimation of infinite small (large) amounts. Fang et al. gave examples of its application techniques [3]. In 2005, Wang et al. gave Taylor's formula with the aggregate balance and proved it with the famous Newton-Leibniz formula, while drawing some conclusions [4]. In 2007, Aljinović et al. used a modified variant of Taylor's formula to demonstrate a fresh generalization of the weighted majorization theorem for n-convex functions [5]. In 2000, Zhao et al. used the Taylor’s theorem to estimate the impact of the land layer parameter randomness on the field transmission function [6]. In 2007, Based on the analysis of Taylor's formula and its revised form, by selecting the production data of metal mines typically mined in 36 different periods of China, the argumental analysis showed that the use of these formulas to calculate the scale of the production of metallic mines was not satisfactory, and the formulas were re-adjusted by Minggui Zheng et al. while establishing a measurement economic theoretical model taking into account integrated factors such as technological progress, and finally using examples, the calculation was made and the formula's use was explained [7]. In 2008, due to the complexity of the high-speed rotation ballistic equation itself, plus the size, direction and point of action of the modification force will affect the rotational movement characteristics of the bullet, the correction may cause intense movement of the ballistic texture. These facts make it difficult to determine the force of modification by calculating the ballistics. For this purpose, Huquan Li et al. proposed a model based on Taylor's formula for the rapid calculation of the actual ballistic modification [8].

In section 2, the paper talks about the proof of the Taylor's theorem and its two different kinds of remainders. Two examples of the applications of the theorem are also given. In section 3, the proof of the Taylor's theorem in several variable functions is given. Its remainder is also mentioned.

2. Taylor Mean Value Theorem

2.1. Derivation of Taylor's theorem in unary functions

In order to make research easier, people frequently aim to employ some simple functions to approximate the expression for some more complex functions. Polynomials are utilized to approximate the expression function since they can determine the value of an independent variable's function as long as it can be calculated using finite arithmetic addition, subtraction, and multiplication. When $|x|$ is small, the following approximations are known in differential applications:

$$e^y \approx 1 + y \quad \ln(1+y) \approx y. \tag{1}$$

These are illustrations of one-time polynomials used to approximate expression functions. These values of the major polynomials and their first derivatives at x=0 is obviously equivalent to the corresponding values of the approximation functions and their derivatives. However, this approximation's precision is just around the upper order infinitesimal size of x, hence it is not very accurate. Naturally, higher polynomials were used to approximate the function in order to increase accuracy. As a result, the following assumptions are made:

Theorem 2.1 (Taylor Mean Value Theorem 1, [9]) If the function g(x) has an n derivative, then the neighborhood of x exists. For any x in the neighborhood, there is

$$g(x) = g(x_0) + g'(x_0)(x - x_0) + \frac{g''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{g^{(n)}(x_0)}{n!}(x - x_0)^n + R_n(x), \tag{2}$$

And

$$R_n(x) = o((x - x_0)^n). \tag{3}$$

Proof: Set $g(x)$ in the x, with an n derivative and try to find a question about the $(x - x_0)$ Polynomial of n times.

$$p_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n. \tag{4}$$

To estimate the function $g(x)$, the difference between $p(x)$ and $g(x)$ when $x \to x_0$ needs to be smaller than $(x - x_0)^n$ higher order infinity. Let's say the value of the function $p(x)$ in x_0 and its derivative up to n in x_0 is equal to $g(x_0)$, $g'(x_0)$,⋯,$g^{(n)}(x_0)$ so $p_n(x_0) = g(x_0)$, $p_n'(x_0) = g'(x_0)$,⋯, $p_n^{(n)}(x_0) = g^{(n)}(x_0)$. Use these equations to determine the coefficient of the polynomial a_0, a_1,⋯, a_n. So $a_0 = g(x_0)$, $1 \cdot a_1 = g'(x_0)$, $2!\, a_2 = g''(x_0)$,⋯, $n!\, a_n = g^{(n)}(x_0)$. Thus $a_0 = g(x_0)$, $a_1 = g'(x_0)$, $a_2 = \frac{1}{2!}g''(x_0)$,⋯, $a_n = \frac{1}{n!}g^{(n)}(x_0)$. Put the coefficients into the polynomial, a function will appear:

$$p_n(x) = g(x_0) + g'(x_0)(x - x_0) + \frac{g''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{g^{(n)}(x_0)}{n!}(x - x_0)^n. \tag{5}$$

The following theorem states that the polynomial (5) is the n-times polynomial needed. Taylor Mean Value Theorem 1 says if the function $g(x)$ has an n derivative at x_0, then the neighborhood of x exists. For any x in the neighborhood, there is

$$g(x) = p_n(x) + R_n(x), \tag{6}$$

And

$$R_n(x) = o((x - x_0)^n). \tag{7}$$

The polynomial (5) is called the function $g(x)$ at x_0 of n times Taylor polynomials, formula (6) is called $g(x)$ at x_0 of the order n Taylor formula with the Peano's form of remainder $R_n(x)$. The expression (7) is called the Peano's form of remainder, which is the error caused by using Taylor polynomials to approximate the $g(x)$. However, it cannot specifically estimate the size of the error. Taylor's theorem, with another form of remainder, solves this problem.

Theorem 2.2 (Taylor Mean Value Theorem 2, [9]) If the function $g(x)$ has an n+1 derivative in the neighborhood $U(a)$ belonging to a. Then for all $x \in U(a)$, there is:

$$g(x) = \sum_{k=1}^{n} \frac{g^{(k)}(a)}{k!}(x-a)^k + R_n(x), \tag{8}$$

With

$$R_n(x) = \frac{g^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}, \tag{9}$$

ξ is a value between a and x.

Remark 2.1 Equation (8) is called $g(x)$ of nth-order Taylor formula with a Lagrangian term at x_0, and the expression of $R_n(x)$ is called the Lagrange term. When n=0, the Taylor formula (8) becomes the Lagrange mean value formula:

$$g(x) = g(x_0) + g'(\xi)(x - x_0)(\xi \text{ is a value between } x_0 \text{ and } x). \tag{10}$$

Remark 2.2 From the Taylor Mean Value theorem 2, people know that when using polynomial $p_n(x)$ to approximate function $g(x)$, the error is $|R_n(x)|$. For a fixed n, when $x \in U(x_0)$, $\left|g^{(n+1)}(x)\right| \le M$, then there is an estimation:

$$|R_n(x)| = \left|\frac{g^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}\right| \le \frac{M}{(n+1)!}|x-x_0|^{n+1}. \tag{11}$$

Remark 2.3 In Taylor's formula (6), if $x_0 = 0$, then there is a Maclaurin's theorem with the Peano's form of remainder

$$g(x) = g(0) + g'(0)x + \frac{g''(0)}{2!}x^2 + \cdots + \frac{g^{(n)}(0)}{n!}x^n + o(x^n). \tag{12}$$

In Taylor's formula (8), if $x_0 = 0$, then ξ is between 0 and x. So $\xi = \theta x\ (0 < \theta < 1)$ can be set, then the Taylor's formula (8) becomes a simpler form, the so-called Maclaurin's theorem with Lagrange's form of remainder.

$$g(x) = g(0) + g'(0)x + \frac{g''(0)}{2!}x^2 + \cdots + \frac{g^{(n)}(0)}{n!}x^n + \frac{g^{(n+1)}(\theta x)}{(n+1)!}x^{n+1} \quad (0 < \theta < 1). \tag{13}$$

The estimation formula from (12) or (13) can be found:

$$g(x) \approx g(0) + g'(0)x + \frac{g''(0)}{2!}x^2 + \cdots + \frac{g^{(n)}(0)}{n!}x^n. \tag{14}$$

The error estimation formula (11) will turn into:

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x|^{n+1}. \tag{15}$$

2.2. Applications of Taylor's theorem in unary functions

Example 2.1: Find the nth-order Maclaurin's formula with Lagrange's form of remainder of function $g(x) = e^x$.

Proof:

$$g'(x) = g''(x) = \cdots = g^{(n)}(x) = e^x, \tag{16}$$

$$g(0) = g'(0) = g''(0) = \cdots = g^{(n)}(0) = 1. \tag{17}$$

Put these values into the formula (13) and the following formula will appear:

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{e^{\theta x}}{(n+1)!}x^{n+1},\ (0 < \theta < 1). \tag{18}$$

If e^x is expressed by the nth-order Maclaurin's theorem:

$$e^x \approx 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}. \tag{19}$$

The error will be:

$$|R_n(x)| = \left|\frac{e^{\theta x}}{(n+1)!}x^{n+1}\right| < \frac{e^{|x|}}{(n+1)!}|x|^{n+1}\ \ (0 < \theta < 1). \tag{20}$$

If $x = 1$, the estimation of e will be:

$$e \approx 1 + 1 + \frac{1^2}{2!} + \cdots + \frac{1^n}{n!}. \tag{21}$$

Its error will be:

$$|R_n| < \frac{e}{(n+1)!} < \frac{3}{(n+1)!}. \tag{22}$$

When $n = 10$, $e \approx 2.718\,282$, the error is smaller than 10^{-6}.

Example 2.2: Using the Taylor's formula with the remainder of the Peano's form to find the value of $\lim_{x \to 0} \frac{\sin x - x\cos x}{\sin^3 x}$.

Proof: Due to the denominator of the fractions $\sin^3 x \sim x^3\ (x \to 0)$, only $\sin x$ and $x\cos x$ need to be expressed in Peano's form of remainder:

$$\sin x = x - \frac{x^3}{3!} + o(x^3). \tag{23}$$

$$x\cos x = x - \frac{x^3}{2!} + o(x^3). \tag{24}$$

Therefore,

$$sin\,x - x\,cos\,x = x - \frac{x^3}{3!} + o(x^3) - x - \frac{x^3}{2!} + o(x^3) = \frac{1}{3}x^3 + o(x^3). \quad (25)$$

$$\lim_{x\to 0}\frac{sin\,x - x\,cos\,x}{sin^3 x} = \lim_{x\to 0}\frac{\frac{1}{3}x^3 + o(x^3)}{x^3} = \frac{1}{3}. \quad (26)$$

3. Derivation of the Taylor's theorem in multivariate functions

Creating an auxiliary function, turning the binary function into a unary function, applying the well-known Taylor formula to the unary function, and using the differential method to the compound function are the steps in the procedure for getting the Taylor formula for the binary function.

Let $u(t) = g(x_0 + th, y_0 + tk)$, $0 \le t \le 1$. So $u(t) = g(x, y)$, $x = x_0 + th$, $y = y_0 + tk$, $0 \le t \le 1$. Then $u(0) = g(x_0, y_0)$, $u(1) = g(x_0 + h, y_0 + k)$. So, the Taylor's formula of the function $g(x_0 + h, y_0 + k)$ at the point $P(x_0, y_0)$ is the Taylor's formula (Maclaurin's theorem) at the point 0 when $t = 1$ of the unary function $u(t)$.

$$u(1) = u(0) + u'(0) + \frac{u''(0)}{2!} + \cdots + \frac{u^{(n)}(0)}{n!} + \frac{u^{(n+1)}(\theta)}{(n+1)!}\ (0 < \theta < 1). \quad (27)$$

The function (2) can be turned into another function:

$$\begin{aligned} & g(x_0 + h, y_0 + k) \\ &= g(x_0, y_0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)g(x_0, y_0) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 g(x_0, y_0) + \cdots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n g(x_0, y_0) + \\ & \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} g(x_0 + \theta h, y_0 + \theta k)\ (0 < \theta < 1). \end{aligned} \quad (28)$$

$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^p g(x, y)$ can be written in the form of $\sum_{r=0}^{p} C_p^r\, h^{p-r} k^r \frac{\partial^p f}{\partial x^{p-r} y^r}$. The function (28) is called the Taylor's theorem of the binary function $g(x, y)$ at the point $P(x_0, y_0)$. By using a similar method, Taylor's theorem of the function with several variables can be found.

Theorem 3.1([10]) If D is an open set in R^n, $a = (a_1, a_2, \cdots, a_n)$, $h = (h_1, h_2, \cdots, h_n)$ and $[a, a + h] \subset D$. If g has an $m + 1$ partial derivative in the field of point a, the formula (29) will exist.

$$\begin{aligned} & g(a_1 + h_1, a_2 + h_2, \cdots, a_n + h_n) \\ &= g(a_1, a_2, \cdots, a_n) + \sum_{i=1}^{m-1}\frac{1}{i!}\left(h_1\frac{\partial}{\partial x_1} + h_2\frac{\partial}{\partial x_2} + \cdots + h_n\frac{\partial}{\partial x_n}\right)^i g(a_1, a_2, \cdots, a_n) \\ & + \frac{1}{m!}\left(h_1\frac{\partial}{\partial x_1} + h_2\frac{\partial}{\partial x_2} + \cdots + h_n\frac{\partial}{\partial x_n}\right)^m g(a_1 + \theta h_1, a_2 + \theta h_2, \cdots, a_n + \theta h_n). \end{aligned} \quad (29)$$

$R_m = \frac{1}{m!}\left(h_1\frac{\partial}{\partial x_1} + h_2\frac{\partial}{\partial x_2} + \cdots + h_n\frac{\partial}{\partial x_n}\right)^m g(a_1 + \theta h_1, a_2 + \theta h_2, \cdots, a_n + \theta h_n)$ is called the Lagrange's form of remainder.

4. Conclusion

The Taylor's theorem, its evolution, and the state of current study are the primary topics of this essay. The essay explains the relevance and establishment of Taylor's theorem. Two different types of residuals and their use in particular scenarios are presented. Then, Taylor's Theorem in a different version for functions with many variables is derived. The Taylor expansion of multivariate functions can be used to approximate the value of a fixed function. For example, if the function's value at the certain point needs to be calculated, but the function is complex and difficult to calculate directly, Taylor theorem can be used to approximate the calculation. Specifically, one can expand this function into an infinite series at this point, and then calculate only the first few terms to obtain a more accurate approximation. Taylor's formula is essentially a qualitative complexity, which unifies complex operations into algebra plus minus multiplication operations. Thus, the Taylor formula can convert the operations themselves into quantitative complexities and make estimates.

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