The Application of Taylor formula in Limits and Approximation

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Abstract. In higher mathematics, the extremely significant content is Taylor's formula. Simple polynomial functions can be translated from complex functions by this formula. The Taylor formula can be seen as a useful tool to analyze and study many problems in mathematics because of its ability to reduce complexity. The Taylor Formula is an essential mathematical process for solving some problems about limits and approximation and has a unique advantage in rough calculation. The Taylor formula is an important tool in calculus, as it can transform nonlinear problems into linear ones with high accuracy. The topic of this paper is researching the application of the Taylor formula in different fields. In this paper, it will talk about the introduction of the Taylor formula, the expansion of the Taylor formula, and its remainders. It also would talk about the application of the Taylor formula in limits and approximation. As a broad mathematical formulation, the Taylor formula can be used to approximate function singularities, non-analytic behavior, or to simplify complex functions. It can be used in a variety of disciplines, including engineering, physics, and mathematics.

Keywords: Taylor formula, limits, approximation.

1. Introduction

Mathematical analysis [1], also known as advanced calculus, is the oldest and most fundamental branch of analysis in mathematics. It generally refers to the study of the general theory of calculus and infinite series, including underlying theories such as real numbers, functions, and limits. It is a comprehensive subject and is a foundational course for college mathematics majors. Analysis is a branch of mathematics devoted to the study of real and complex numbers and their functions. Its development began with calculus and extended to various properties of functions, such as continuity, differentiability, and integrability. These properties can be used to analyze study and discover patterns in the physical world.

The paper focuses on the Taylor series. In 1715, Taylor first gived the definition of the Taylor Series through Taylor's formula. The Taylor series is a mathematical function represented as an infinite summation, known as a series, and can be represented as a series of terms increasing in power of the independent variable. It arises from the function's derivative at the zero point of the independent variable and is alternatively referred to as the Maclaurin series. Applications of the Taylor series gradients appear not only in the field of mathematics but also in physics and engineering [2], [3]. For example, when a nonlinear problem arises, Taylor's formula can be applied to construct a locally effective linearized model that approximates the original function and simulates the computation using the data from this approximate model to find the solution more conveniently. As the field of mathematics continues to grow, the importance and extensiveness of the Taylor series are gradually realized in research and applications.

2. Preliminary Knowledge

2.1. Taylor Formula

A power series, whose term is a power of x multiplied by a separate constant, can also be used to refer to the Taylor series:

$$g(x) = m_0x^0 + m_1x^1 + m_2x^2 \dots m_nx^n, \tag{1}$$

Where $m_0, m_{1,}, \dots, m_n$ are determined by the derivatives of the function. For instance, the constant m_6 is based on the sixth derivative of the function. The formula would be according to the Lagrange's mean value theorem:

$$\int_m^x g'(x) \cdot \Delta x = g'(z) \cdot (x - m). \tag{2}$$

Therefore:

$$g^{(n)}(z) \cdot (x - m) = g^{(n-1)}(x) - g^{(n-1)}. \tag{3}$$

If integrating the left part, it would be:

$$\int_m^x g^{(n)}(z) \cdot (x - m) \cdot \Delta x = g^{(n)}(z) \int_m^x (x - m) = g^{(n)}(z) \frac{(x-m)^2}{2}. \tag{4}$$

If integrating the right part, it would be:

$$\int_m^x g^{(n-1)}(x) \cdot \Delta x - \int_m^x g^{(n-1)}(m) \cdot \Delta x = [g^{(n-2)}(x) - g^{(n-2)}(m)] - g^{(n-1)}(m)(x - m). \tag{5}$$

Combining the two results gives that:

$$g^{(n)}(z) \frac{(x-m)^2}{2} = g^{(n-2)}(x) - g^{(n-1)}(m)(x - m). \tag{6}$$

$g^{(n)}(x)$ is a constant in this function and equals the n-th derivative calculated at point z between m and x.

All of them indicate that the pattern should be obvious. Integrating n times in succession to ultimately find $g_n(x)$, the remarkable conclusion would be:

$$g^{(n+1)}(z) \frac{(x-m)^{n+1}}{(n+1)!} = g(x) - g(m) - g'(m) \frac{(x-m)}{1!} - g''(m) \frac{(x-m)^2}{2!} - \dots - g^{(n)}(m) \frac{(x-m)^{(n)}}{(n)!}. \tag{7}$$

Which equals to:

$$g(x) = g(x_0) + g'(x_0)(x - x_0) + \frac{g''(x_0)}{2!}(x - x_0)^2 + \dots + \frac{g^{(n)}(x_0)}{n!}(x - x_0)^n + g^{(n+1)}(z) \frac{(x-m)^{n+1}}{(n+1)!}, \tag{8}$$

And can simplifies to:

$$g(x) = \sum_{k=0}^{n} g^{(k)}(m) \frac{(x-m)^k}{k!} + g^{(n+1)}(z) \frac{(x-m)^{n+1}}{(n+1)!}. \tag{9}$$

If using $p_n(x)$ to approximate the value of $g(x)$, $R_n(x)$ would be the remainder of $g(x)$. Therefore, $p_n(x) = \sum_{k=0}^{n} g^{(k)}(m) \frac{(x-m)^k}{k!}$ and $R_n(x) = g^{(n+1)}(z) \frac{(x-m)^{n+1}}{(n+1)!}$.

The function would be:

$$g(x) = p_n(x) + R_n(x). \tag{10}$$

The definition of the Taylor Formula is that if the polynomial $g(x)$ is in an open interval (p,q), which contains x_0, until the (n+1)-th derivative, then for $\forall x \in (p, q)$:

$$g(x) = g(x_0) + g'(x_0)(x - x_0) + \frac{g''(x_0)}{2!}(x - x_0)^2 + \dots + \frac{g^{(n)}(x_0)}{n!}(x - x_0)^n + R_n(x). \tag{11}$$

2.2. Expansion of the Taylor Formula

If x_0 is equal to 0, the Maclaurin's series would be:

$$g(x) = g(0) + g'(0)x + \frac{g''(0)}{2!}x^2 + \ldots + \frac{g^{(n)}(0)}{n!}x^n + \frac{g^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}(0 < \theta < 1). \tag{12}$$

According to Maclaurin's series, there are some special expansions:

$$e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^k}{k!} + \ldots \tag{13}$$

$$\sin x = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^{2k-1}}{(2k-1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + (-1)^{k+1}\frac{x^{2k-1}}{(2k-1)!} + \ldots \tag{14}$$

$$\cos x = \sum_{k=0}^{\infty}(-1)^n\frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + (-1)^k\frac{x^{2k-1}}{(2k)!} + \ldots \tag{15}$$

$$\ln(1 + x) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots + (-1)^{k+1}\frac{x^k}{k} + \ldots \tag{16}$$

2.3. Remainder

There are two kinds of remainders.

First, according to 2.1, the remainder is $R_n(x) = g^{(n+1)}(z)\frac{(x-m)^{n+1}}{(n+1)!}$(c is a constant between x and m, which is known as the Lagrange remainder.

Second, according to Taylor's mean value theorem, if the polynomial $p_n(x)$ approximately express the function $g(x)$, its error is $|R_n(x)|$. For a fixed n, when $x \in (p, q)$, $|g^{(n+1)}(x)| \leq M$, there is an estimator:

$$|R_n(x)| = |\frac{g^{(n+1)}(z)}{(n+1)!}(x - x_0)^{n+1}| \leq \frac{M}{(n+1)!}|x - x_0|^{n+1}. \tag{17}$$

Then:

$$\lim_{x \to x_0}\frac{R_n(x)}{(x-x_0)^n} = 0. \tag{18}$$

Thus, it can be seen, when $x \to x_0$, the error of $|R_n(x)|$ is infinitesimal than the higher order of $(x - x_0)^n$, which is:

$$R_n(x) = o[(x - x_0)^n]. \tag{19}$$

This formula of $R_n(x)$ is known as the Peano remainder.

3. Applications of Taylor Series in Limits

In mathematics, the term limits refer to the behavior of a function as it approaches a point, rather than precisely at that point. During the calculation of limits, there are two values involved, as the function gets closer to the dependent variable corresponding to the independent variable [4]. Limits are fundamental, especially in limits and mathematical analysis. Without understanding limits, students can hardly understand much in calculus. The reason is that limits are used to define the continuity, derivatives, and integrals of functions [5].

The expression for the limits is as follows:

$$\lim_{a \to a_0} f(a) = b. \tag{20}$$

This statement that f (a) has a limit b as an approach a_0, also signifies that point b = f(a) can be brought as close as possible to point b by choosing point a near enough a_0, while still being different from it.

Furthermore, in this case, the limits of a function f(a) would only exist [6]: $\lim_{a\to a_0} f(a) = b$. If and only if:

$$\lim_{a\to a_0^+} f(a) = \lim_{a\to a_0^-} f(a) = b. \tag{21}$$

Formulas are essential tools for limit computation. Firstly, knowing the fundamental limit laws, along with the limit laws involving exponents and roots, and the limit laws involving arithmetic operations enable us to conduct basic operations like addition, subtraction, multiplication, and division on functions for the sake of finding the limits. Secondly, L'Hôpital's rule is introduced when a function cannot be solved independently, precisely this rule allows the evaluation of limits when dealing with indeterminate forms using derivatives. Frequently, this rule is referred to as Bernoulli's rule, as its origin, although initially published by Hospital, can be traced back to 1694 and is credited to Bernoulli [7].

By taking the limit of different functions, some of the functions will end up with the following so-called the Indeterminate form:

$$\frac{0}{0}, \frac{\pm\infty}{\pm\infty}, \infty-\infty, 0\times\infty, 0^0, 1^\infty, \infty^0. \tag{22}$$

The determinate form is expressed as follows:

$$\infty+\infty, -\infty-\infty, 0^\infty = 0, 0^{-\infty} = -\infty, \infty\times\infty. \tag{23}$$

L'Hôpital's rule stipulates that when substituting a limit directly results in an indeterminate form, the required step is to differentiate both the denominator and the numerator in order to calculate the limit.

Firstly, the direct substitution of a limit results in an indeterminate form is the following:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{0}{0}, \frac{\pm\infty}{\pm\infty}. \tag{24}$$

Following that, deriving the function results in:

$$\lim_{x\to a}\frac{f\prime(x)}{g\prime(x)}. \tag{25}$$

Lastly, we can compute the limit by substitution.

Example 3.1: Please determine the limit for the following function:

$$\lim_{x\to -2}\frac{x+2}{x^2+3x+2} . \tag{26}$$

Proof: When attempting to evaluate the limit by substituting -2, the direct substitution method is ineffective. Instead, the application of L'Hôpital's rule is warranted. Firstly, differentiating the function follows. Then by substituting the number -2, it becomes $\lim_{x\to -2}\frac{1}{2(-2)+3} = -\frac{1}{1} = -1$. Therefore, this results in a limit of -1 for the function.

Example 3.2: Please determine the limit for the following function:

$$\lim_{x\to 0}\frac{x^2}{\cos -1}. \tag{27}$$

Proof: When attempting to evaluate the limit by substituting 0 as $\lim_{x\to 0}\frac{x^2}{\cos -1} = \frac{0}{0}$, the direct substitution method proves ineffective. Instead, the application of L'Hôpital's rule is warranted. Firstly, apply L'Hôpital's rule to differentiate the function, $\lim_{x\to 0}\frac{2x}{-\sin x} = \frac{0}{0}$. However, sometimes it may be necessary to apply L'Hôpital's rule multiple times, necessitating multiple derivations of the function. Taking the second derivative of the function $\lim_{x\to 0}\frac{2}{-\cos x} = -2$. In this case, the limit of the function can consequently be determined to be -2.

$$\lim_{x\to 0}\frac{x^2}{\cos -1} = -2 \tag{28}$$

Therefore, the two conditions for applying the L'Hôpital's rule are an indeterminate form, and the function must be analytic.

However, situations arise where the indeterminate form emerges after the first direct substitution, and despite repeated applications of L'Hôpital's rule, the indeterminate form persists. This is the reason for using the Taylor Series becomes necessary.

Example 3.3: Please evaluate the limit for the function provided below:

$$\lim_{x\to\infty}\frac{(\ln x)^2}{e^{2x}}. \tag{29}$$

Proof: When attempting to evaluate the limit by substituting ∞, the substitution method proves ineffective, $\lim_{x\to\infty}\frac{(\ln x)^2}{e^{2x}} = \frac{\infty}{\infty}$. Instead, the application of L'Hôpital's rule is warranted. Firstly, apply L'Hôpital's rule to differentiate the function, $\lim_{x\to\infty}\frac{2(\ln x)}{x2e^{2x}} = \frac{\infty}{\infty}$. Keep applying L'Hôpital's rule multiple times, $\lim_{x\to\infty}\frac{1}{2x^2e^{2x}} = 0$, which means keep taking multiple derivatives of the function. Therefore, the limit of the function can be determined to be 0.

$$\lim_{x\to\infty}\frac{(\ln x)^2}{e^{2x}} = 0. \tag{30}$$

The Taylor Series can be expressed as follows:

$$f(x) = \sum_{k=0}^{n} f^{(k)}(a)\frac{(x-a)^k}{k!} + f^{(n+1)}(c)\frac{(x-a)^{n+1}}{(n+1)!}. \tag{31}$$

The expression for Maclaurin’s series can be written as follows:

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots. \tag{32}$$

Using the Taylor Series can assist in expanding the initially unsolvable problem into a polynomial power series. Subsequently, by employing L'Hôpital's rule or other methods to obtain the constant term, one can determine the limits of a function.

When individuals apply the Taylor Series to expand a function, the process of function expansion becomes crucial. It is essential to initially identify the distinct functions within the function and carefully expand each function using the Taylor Series (or Maclaurin’s series). Once the function becomes analytic, various methods can be employed to ascertain the function's limits.

Example 3.4: Please determine the limit for the following function:

$$\lim_{x\to 0}\frac{\cos(2x)-1}{x^2e^x}. \tag{33}$$

Proof: When attempting to evaluate the limit by substituting 0, the direct substitution method proves ineffective, $\lim_{x\to 0}\frac{\cos(2x)-1}{x^2e^x} = \frac{0}{0}$. Instead, the application of L'Hôpital's rule is warranted. Firstly, use L'Hôpital's rule to take the derivative of the function. However, in this case, after applying L'Hôpital's rule multiple times, $\lim_{x\to 0}\frac{-2\sin(2x)}{2xe^x+x^2e^x} = \frac{-2}{0}$, the function still ends up in an indeterminate form. Therefore, the Taylor series needs to be used to expand the functions individually. There are two functions that need to be expanded:

$$\cos 2x = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n!(2x)^{2n}} \tag{34}$$

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} \tag{35}$$

After the expansion the expression becomes $\lim_{x\to 0}\frac{\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{2n!(2x)^{2n}}\right)-1}{x^2\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}\right)}$, which is equivalent to $\lim_{x\to 0}\frac{\left(1+\frac{-1}{2}2^2x^2+\frac{1}{24}2^4x^4+\ldots\right)-1}{\sum_{n=0}^{\infty}\frac{x^{n+2}}{n!}}$. By applying the L'Hôpital's rule for multiple times, first derivative, $\lim_{x\to 0}\frac{\left(-2x^2+\frac{2}{3}x^4+\ldots\right)}{(x^2+x^3+\frac{x^4}{2}+\ldots)}$, second derivative $\lim_{x\to 0}\frac{\left(-4x+\frac{8}{3}x^3+\ldots\right)}{(3x+3x^2+2x^3+\ldots)}$, third derivative and finally third derivative, $\lim_{x\to 0}\frac{(-4+8x^2+\ldots)}{(2+6x+6x^2+\ldots)}$. Ultimately, the limit of this function is determined to be -2. Therefore,

$$\lim_{x\to 0}\frac{(-4+8x^2+\ldots)}{(2+6x+6x^2\ldots)} = \frac{-4}{2} = -2. \tag{36}$$

Example 3.5: Please determine the limit for the following function:

$$\lim_{x\to 0}\frac{\cos(x^2)-e^{x^4}}{\sin x^4}. \tag{37}$$

Proof: When attempting to evaluate the limit by substituting 0, the direct substitution method proves ineffective, $\lim_{x\to 0}\frac{\cos(x^2)-e^{x^4}}{\sin x^4} = \frac{0}{0}$. Instead, the application of L'Hôpital's rule is warranted. However, after applying L'Hôpital's rule multiple times, the function still ends up in an indeterminate form. Therefore, the Taylor series needs to be used to expand the functions individually. There are three functions need to be expanded:

$$\cos x^2 = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n!}(x)^{4n} = 1 + \frac{x^4}{2} + \frac{x^8}{24} + \cdots \tag{38}$$

$$e^{x^4} = \sum_{n=0}^{\infty}\frac{x^{4n}}{n!} = 1 + x^4 + \frac{x^8}{2} + \ldots \tag{39}$$

$$\sin x^4 = \sum_{n=0}^{\infty}\frac{(-1)^{2n+1}}{(2n+1)!}(x)^{8n+4} = x^4 - \frac{x^{12}}{6} + \frac{x^{20}}{120} + \cdots \tag{40}$$

In this case, there are two methods for computing the limits after the Taylor expansion. The first, a familiar method, involves taking multiple derivatives one by one using L'Hôpital's rule. However, in this case, it becomes quite complex. As demonstrated here, the fourth derivative of the function must be computed to reach the solution: the first derivative $\lim_{x\to 0}\frac{-6x^3+\ldots}{4x^3+.\,.\,.}$, the second derivative

$\lim_{x\to 0}\frac{-18x^2+\dots}{12x^2+...}$, the third derivative $\lim_{x\to 0}\frac{-36x+\dots}{24x+...}$, and finally the fourth derivative $\lim_{x\to 0}\frac{-36+\dots}{24+...}$. Ultimately, the limit is found to be $\lim_{x\to 0}\frac{-\frac{3}{2}+\frac{-11}{24}x^4+\dots}{1-\frac{x^8}{6}+...}=\frac{-3}{2}$.

The alternative method is to identify and take the common factor of x^4, which simplifies the process significantly to $\lim_{x\to 0}\frac{-\frac{3}{2}+\frac{-11}{24}x^4+\dots}{1-\frac{x^8}{6}+...}$. In the end, both approaches to finding the limits converge to the same result, where the limit of the function is $\frac{-3}{2}$.

To sum up, all these formulas are essential tools for determining the limits of functions. They help in enhancing people's comprehension of a function's behavior and in solving various types of limit problems. Particularly, when seeking to find a function's limits, the key is to isolate the constant term, thereby simplifying the function is crucial. Hence, when finding the limits of a function, it is necessary to use the formulas appropriately and wisely in order to assist people in analyzing and differentiating the problem when dealing with various situations.

4. Applications of Taylor Series in Approximation

For indeterminate Taylor formula applications [8], the Taylor series would approximate the original function. When computing the value of a function at a particular point, sometimes the function does not have a convenient parsing expression for accurate computation. This may be because the function itself is complex, or because finding a simple mathematical formula to describe it is hard. In this case, using Taylor's formula is allowed [9], because it is a way to represent a function as a power series. Also, by expanding the function polynomially around a given point, obtaining an expression for the series containing an infinite number of polynomials is achievable.

Taylor series are useful for generalizing the function around a specific point by truncating the series at a certain term [10]. The Taylor series of function f(a) centered at b is:

$$f(a)\approx f(b)+f'(b)(a-b)+\frac{f''(b)(a-b)\cdot 2}{2!}+\frac{f'''(b)(a-b)\cdot 3}{3!}+\dots \tag{41}$$

The Remainder or Error term in the Taylor series provides a measure of how well the series approximation matches the original function. Such as the Lagrange form of the remainder (c is between a and x, and n represents the term's number used in the Taylor series):

$$Rn(a) = f^{(n+1)}(c)\frac{(a-b)^{n+1}}{(n+1)!}. \tag{42}$$

Example 4.1: The top three terms of the Taylor series expansion are used for exploiting $f(a)=\sqrt[3]{a}$ centered at a = 8, approximate $\sqrt[3]{8.1}$:

Proof: f(a) can be write as

$$f(a) = \sqrt[3]{a}\approx 2+\frac{a-8}{12}-\frac{(a-8)^2}{288} . \tag{43}$$

The first three terms of the Taylor series are sufficient to eatimate $\sqrt[3]{a}$. Computing the three terms of a=8.1 leads to that:

$$f(a)=\sqrt[3]{8.1}\approx 2+\frac{8.1-8}{12}-\frac{(8.1-8)^2}{288}=2.008298611\dots \tag{44}$$

The formula above is accurate to six decimal places for $\sqrt[3]{8.1}$ with only three terms.

5. Conclusion

In conclusion, the paper has discussed the Taylor Series, which is one of the crucial formulas in mathematics. Taylor Series can be used to approximate a function around a specific point and can be used to solve various mathematical problems such as derivatives, integrals. Firstly, the paper discusses the core of the Taylor Series, which serves as a powerful tool to simplify the complexity of a function when dealing with complex functions. As the n-th Taylor polynomial increases, it becomes more accurate in approximating the original function. Secondly, in theory, the Taylor series is an infinite sum of polynomial terms. However, in practice, people only expand the function into a finite number of terms, and the remainder represents the difference between the actual function and the approximation. There are two types of remainders, Lagrange and Peano remainder. Lastly, when finding the limits of a function, by using the Taylor Series, people can successively solve the originally indeterminate function, which can't be solved using L'Hôpital's rule. This is done by using the Taylor series to approximate the original function. Then the problem can be solved by using the combination of different formulas accordingly. In essence, the Taylor Series serves as a fundamental formula in calculus and is crucial in academia and practice. It has helped not only mathematicians but also scientists and engineers to understand approximation better and solve various functions by estimating how a function looks. Thus, Taylor Series boosts progress and innovation across various fields.

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