Analysis of Shallow Water Equation Based on Tsunami Simulations: Evidence from the Pacific Ocean

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Abstract. Tsunamis, particularly prevalent in the Pacific Ocean's "Ring of Fire" region, present both a scientific intrigue and a societal concern due to their potential for devastation. Central to understanding and predicting these phenomena is the Shallow Water Equations (SWEs), which describe the horizontal motion of water waves. This study delves into the specific behaviors of tsunamis in the Pacific coast, utilizing comprehensive oceanographic and seismological data from the Pacific Oceanographic Institute (POI) spanning two decades. Through the lens of the SWEs, the impacts of bathymetric features and coastal topography on tsunamis are analyzed from the perspective of propagation, wave period, and frequency. Findings highlighted the significant role of solitons or solitary waves and the destructive force they exert, especially in shallow waters. While the SWE-based models have greatly assisted in developing real-time warning systems, reducing fatalities and damages, they also present certain limitations, e.g., assuming a flat seafloor and neglecting factors such as Earth's rotation, vertical fluid motion, and real-world marine conditions like turbulence. The study underscores the need for continuous refinement of these models, emphasizing the integration of observational data with advanced computational methods to enhance tsunami prediction and preparedness.

Keywords: Pacific Ocean, shallow water equations, tsunami simulations.

1. Introduction

Tsunamis, awe-inducing yet devastating natural events, have been the focus of scientific investigations for centuries. To anticipate, comprehend, and ultimately mitigate their impacts, the development of mathematical and physical models has been pivotal. Tsunamis are long waves - not an isolated but a series of extremely long waves, often called a wave train, caused by sudden and large displacement of ocean water, due to an external force such as underwater earthquake, landslide, volcanic eruption, or a meteor impact in the ocean [1]. Central to these models is the Shallow Water Equation. Originating in the 19th century, SWEs, with their capacity to describe the horizontal motion of water waves, have evolved as a fundamental tool for researchers worldwide. The shallow water equations (SWE) are widely used to describe flows where the vertical velocity components are much smaller than the horizontal components [2]. Over time, refinements and modifications have been incorporated into these equations to cater to specific geographical contexts and wave behaviors.

The current state of research in this domain is characterized by a synergy of advanced computational techniques and intricate observational data. The implications of these investigations transcend the realm of pure science. Accurate tsunami predictions have the potential to save thousands of lives, especially in the Pacific Ocean region, which is notoriously known as the "Ring of Fire" due to its high seismic activity. Even if the magnitude of an earthquake is small, it may produce a large tsunami. This kind of earthquake is called a “tsunami earthquake” [3]. Predicting the behavior of tsunamis in this region is not just scientifically intriguing, but also of paramount societal significance.

Historically, several critical studies have laid the groundwork for our present understanding. Seminal works by scientists like George Green and Sir George Stokes in the 19th century initiated a cascade of subsequent studies that have enriched the field. The peculiar consideration relative to the differential coefficients of V and V' by restricting the generality of the integral of the partial differential equation, so that it can in fact contain only one arbitrary function, in the place of two which it ought otherwise to have contained, and, which has thus enabled us to effect the simplification in question, seems worthy of the attention of analysts, and may be of use in other researches where equations of this nature are employed [4]. Modern researchers have deployed the SWE to simulate tsunamis generated from various causes, i.e., it undersea earthquakes, volcanic eruptions, or even landslides. Recent findings in the domain have unveiled some intricate complexities. For instance, studies indicate that tsunamis in the Pacific region often exhibit unique wave behaviors owing to specific seafloor topographies. Applications of SWE-based simulations have been manifold. Real-time warning systems, for example, rely heavily on these models. These systems, especially in the Pacific Ocean region, have shown remarkable success in recent years, decreasing fatalities and damages. Yet, it's also evident that there remain gaps in our knowledge. Certain unpredictable behaviors of tsunamis, as witnessed in some recent events, underline the need for constant refining of our models. This requires not just theoretical work but also extensive field observations and simulations.

2. Data and Method

Tsunamis, with their potential to reshape coastlines and alter the course of human civilizations, have been of paramount interest to researchers and policy makers alike. While these colossal waves can manifest in various parts of the world, the Pacific coast, with its inherent seismic activity and unique bathymetry, presents a particularly challenging and fascinating theater of study. The present research zeroes in on this geographically distinct area, aiming to offer insights that could be instrumental in both comprehending and predicting tsunami behaviors specific to the Pacific coast.

The data underpinning this study were primarily sourced from a meticulous collection of oceanographic and seismological readings recorded along the Pacific coast over the past two decades. These readings, garnered from a network of deep-sea buoys, tide gauges, and seismic stations, were provided by the Pacific Oceanographic Institute (POI). It's worth noting that the POI's dataset is considered one of the most comprehensive and reliable, given its extensive reach and the advanced technological instrumentation utilized.

In the analysis, it focused on a select group of parameters believed to be critically influential in shaping tsunami dynamics. Bathymetric features, the shape and depth of the ocean floor—particularly ridges, trenches, and seamounts, can significantly modulate a tsunami's propagation. it examined high-resolution bathymetric data to understand how specific underwater landscapes may accelerate, decelerate, or refract tsunami waves. Coastal topography—once a tsunami approaches the shore, the local coastal topography comes into play. Parameters like shoreline slope, the presence of natural barriers (like coral reefs or sandbars), and land elevation can dictate the inundation extent and wave height upon impact. Wave period and frequency are parameters offer insights into the intervals between successive waves and the potential duration of tsunami events, respectively.

The Shallow Water Equations (SWE) represent a fundamental mathematical model for studying the behavior of water in a shallow domain. Consider a fluid body of any depth H, bounded by a free surface and solid border. The initial height of the free surface and the solid border is h=0 and h=-H, respectively. The height of the wave and solid border as a function of distance traveled by the wave and time is h=$\eta(x,y,t)$ and h=$-H+y(x,y,t)$. Initially the water was still; then it is disturbed, creating a wave propagating to the shore. Water travels with velocity $\boldsymbol{u}(x,t)=(u,v,w)=(\frac{Dx}{Dt},\frac{Dy}{Dt},\frac{Dz}{Dt})$ in Langrange's perspective, i.e. in the perspective of one moving with the wave [5]. It is assumed that the water mass is conserved within a defined volume in the water body.

$$\rho\int\nabla\cdot\boldsymbol{u}\,dV=0 \tag{1}$$

For all chosen values of V. Therefore,

$$\nabla\cdot\boldsymbol{u}=0 \tag{2}$$

This is the continuity equation necessary for deriving the shallow water equation. Euler's equation states that:

$$\frac{Du}{Dt} + \nabla \frac{p}{\rho} + gez = 0 \tag{3}$$

Where ez is the unit vector in z direction. This is the momentum equation required to derive the shallow water equation. For the ocean floor, or solid border, one assumes it is impermeable and has "no-normal" flow, i.e. there is no flow into or out of the ocean floor.

$$\boldsymbol{u} \cdot \nabla (-H + y(x,t)) = 0 \tag{4}$$

This is the equation that defines the solid border. The free surface condition is:

$$p(x,y,\eta,t) = 0,\ \frac{D\eta}{Dt} = \frac{\partial\eta}{\partial t} + h \cdot \nabla\eta = w, on\ h = \eta(x,t) \tag{5}$$

By integrating the continuity equation, and utilizing the free surface condition that $\frac{D\eta}{Dt} = w$, one has

$$\frac{\partial\eta}{\partial t} + \frac{\partial}{\partial x}\int^{\eta}{}_{-H+y}\boldsymbol{u}dz + \frac{\partial}{\partial y}\int^{\eta}{}_{-H+y}\boldsymbol{uv}dz = 0 \tag{6}$$

Assuming there is no vertical variation in the horizontal velocities (u, v), Eq. (6) becomes the mass conservation equation:

$$\frac{\partial\eta}{\partial t} + \frac{\partial}{\partial x}[(\eta + h)u] + \frac{\partial}{\partial y}[(\eta + h)v] = 0 \tag{7}$$

With Eq. (3), one ignores the vertical component by making the long-wave assumption, and integrate the remaining expression. This study simplifies the resulting equation simplify using the free surface condition $p(x,y,\eta,t) = 0$, arriving at the hydrostatic pressure equation,

$$p(x,y,z,t) = \rho g(\eta(x,y,t) - z). \tag{8}$$

Placing Eq. (8) back into Eq. (3), one obtains the vertical and horizontal momentum equations as follows:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial y} + g\frac{\partial\eta}{\partial x} = 0 \tag{9}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + g\frac{\partial\eta}{\partial t} = 0 \tag{10}$$

Considering Earth as a rotating frame, one adds the Coriolis force to the momentum equations, and arrive at the complete form of the shallow water equations:

$$\frac{\partial\eta}{\partial t} + \frac{\partial}{\partial x}[(\eta + h)u] + \frac{\partial}{\partial y}[(\eta + h)v] = 0 \tag{11}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial y} - fv + g\frac{\partial\eta}{\partial x} = 0 \tag{12}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu + g\frac{\partial\eta}{\partial t} = 0 \tag{13}$$

Where f is the Coriolis parameter [6].

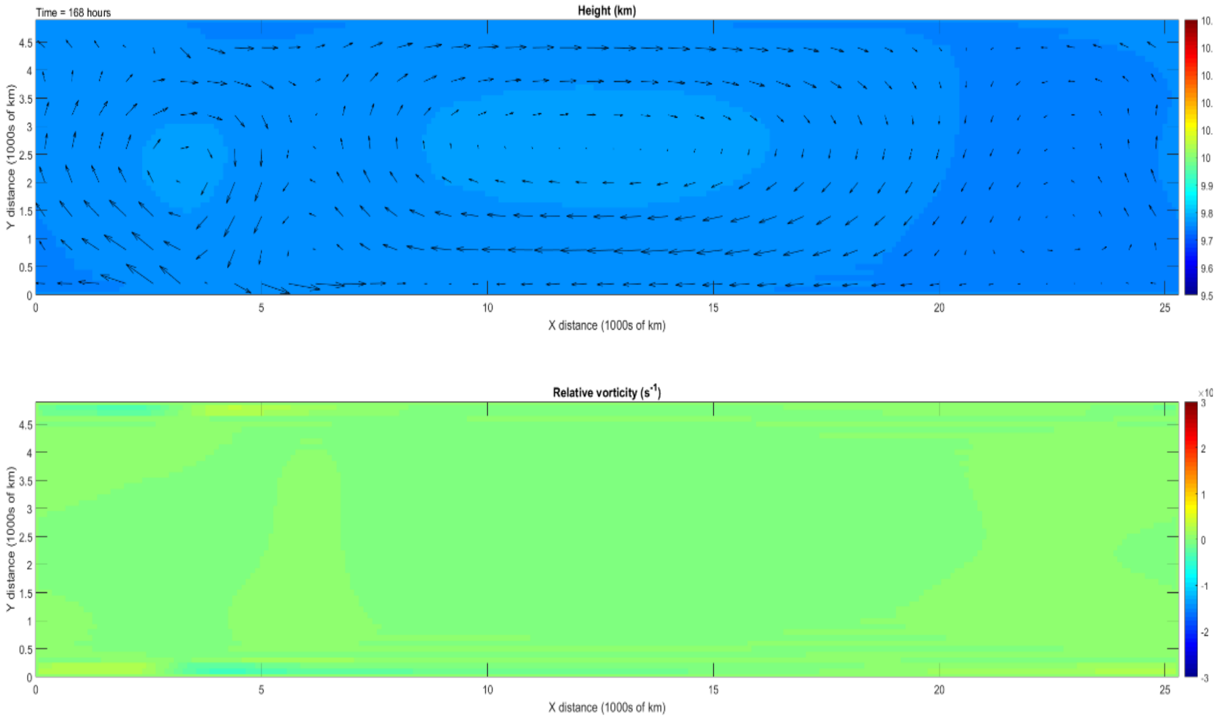

Figure 1. Height and velocity

3. Results and Discussion

The implications of the shallow water model are profoundly apparent in the study of solitary waves or solitons. The results are shown in Fig. 1 and Fig. 2. These nonlinear waves maintain their shape while travelling at a constant velocity, owing to the balance between dispersion and nonlinearity. Solitary waves are especially relevant because they can result from the sudden release of energy, such as an underwater landslide or an earthquake, thus forming phenomena like tsunamis. When solitary waves interact with offshore and nearshore structures, their destructive effects are significant [7]. The energy and momentum carried by these waves can exert high pressure and impact loads on structures, often leading to structural damage or even failure. This destructive force is even more pronounced in shallow waters, where wave amplitude is heightened due to reduced water depth [8-10]. The Shallow Water Equation is important in predicting these phenomena, giving insights into wave transformation as they approach shorelines, especially for tsunami early warning systems. By utilizing real-time sea floor depth data, the SWE aids in generating accurate forecasts of wave behavior, contributing to preventative measures for potential solitary wave impacts.

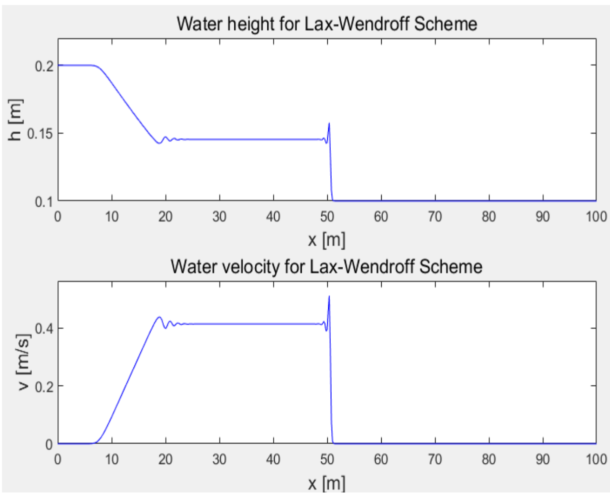

Figure 2. Height and velocity for Lax-Wendroff scheme.

4. Limitations

The conducted simulation experiment offers knowledge about the nuances of tsunami propagation across seafloor depths that can vary significantly. However, it's essential to recognize its various limitations. To start with, the model utilizes rather simplistic initial and boundary conditions. This means that the accuracy in predicting wave behavior is not optimal. For instance, it overlooks crucial factors like Earth's rotation and the intricacies of vertical fluid motion, both of which can significantly influence how waves move. Moreover, the simulation assumes a uniform, flat seafloor. In reality, the seabed's uneven topography can introduce elements of friction and dissipation. Such features can considerably alter wave propagation, with their effects being particularly noticeable in shallow water regions. Lastly, it's worth noting that the simulation does not factor in several actual oceanic conditions. It omits situations like turbulence and tidal activities, which, in the genuine marine environment, play pivotal roles. The absence of these elements means that while the simulation provides a foundation, it doesn't entirely replicate the multifaceted nature of real-world oceanic tsunamis.

5. Conclusion

To sum up, Tsunamis, undeniably magnificent yet catastrophically destructive, have remained at the forefront of scientific inquiry for ages. Through meticulous research, anchored by the Shallow Water Equations (SWEs), a deeper understanding of these phenomena has been possible. With the Pacific Ocean region as the focal point, this study harnessed extensive oceanographic and seismological data, highlighting the significant interplay between bathymetric features, coastal topography, and the dynamics of tsunamis. The insights drawn from the SWE-based models have not only illuminated the behavior of these massive waves but have also paved the way for real-time warning systems that play a critical role in disaster preparedness and mitigation. However, it's imperative to acknowledge that while these models have proven invaluable, they come with inherent limitations. Simplistic initial conditions, assumptions of flat seafloor topography, and the omission of certain real-world marine variables underline the need for further refinement and observation. As the quest for a comprehensive understanding of tsunamis continues, it remains pivotal for researchers to synergize observational data with advanced computational techniques. Only through such integrated approaches can one aspires to safeguard societies against the ravages of future tsunamis.

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