

Mean Value Theorem and Its Uses

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Abstract. The foundation of all mathematical analysis is calculus. The mean value theorem of differential equations and the mean value theorem of integral equations are crucial concepts in calculus. They establish the framework for the entire calculus. This essay discusses three different types of mean value theorems. They have some degree of generalization and proof. Thus, the mean value theorems of Lagrange, Rolle, and Cauchy are all correctly demonstrated in this study. The foundation for these three theorems further establishes the calculus fundamental theorem. They can expand the differential mean value theorem and demonstrate the Newton-Leibniz formula. The application of mean value theorems is discussed in this paper. This research is really important. Because mathematics uses the most fundamental and significant tool used in human investigation. Like the progression from the mean value theorem of the differential to the mean value theorem of the integral, from the known to the unknown.

Keywords: Rolle Mean Value Theorem, Lagrange Mean Value Theorem, Cauchy Mean Value Theorem.

1. Introduction

Analysis is a discipline about limits, continuous functions, and associated theories, including differentiation, measure, integration, analytical functions, infinite sequences and series. The context of real and complex numbers and functions is typically used when these theories are studied. Calculus, which includes the fundamental ideas and methods of analysis, is where analysis originated. However, analysis can be applied to any space of mathematical objects that has a notion of nearness or particular distances between objects and can be differentiated from geometry [1]. Calculus, whose theoretical underpinning is limiting theory, forms the bulk of mathematical analysis. Real number theory serves as the theoretical foundation for limit theory. Differential calculus and integral calculus are collectively referred to as calculus. The principal applications of early calculus were in astronomy, mechanics, and geometry. Later, people also referred to calculus analysis as infinitesimal analysis, with the latter term explicitly referring to the analysis and solution of computing issues through the application of infinitesimal, infinity, and other limit processes. The differential mean value theorem is a key concept in mathematical analysis. The term "differential mean value theorem" refers to a group of mean value theorems that are effective tools for the study of functions. Lagrange's theorem is the most significant part of the text. Other mean value theorems might be thought of as variations or generalizations of Lagrange's mean value theorems. It is common practice to utilize the mean value theorem of differentiation [2], which represents the connection between the locally of the derivative and the integrity of the function. It can be used to prove inequalities, talk about equation roots, compute limits, talk about series convergence and divergence, and talk about monotonicity of functions, among other things. Additionally, researchers can compute the approximation values and find the extreme value using the monotonicity of the function [3].

The differential mean value theorem serves as the primary focus of this essay. The geometric importance of the differential mean value theorem is clear. According to Lagrange's theorem, the tangent line of a differentiable function at the end of the curve must be parallel to the string. The differential mean value theorem has its roots in the ancient Greek era BC, in this sense. In their geometric studies, the ancient Greek mathematicians arrived to the following conclusion: the tangent line of the parabola bow's vertex must be parallel to the parabola bow's base, which is a specific case of Lagrange's theorem. This result was effectively exploited by Archimedes to determine the size of the parabolic arch. In his Non-Component Geometry, Cavalier provided an intriguing lemma for

handling tangents of solid and flat figures. The same fact is stated by him: there must be a place on a curve segment where the tangents are perpendicular to the curve's threads. This is Cavalier's theorem, the mean value theorem of the differential in geometric form. The eminent French mathematician Fermat published his theorem in 1637 and included it in his method for determining the maximum and minimum values. Typically, it is regarded as the differential mean value theorem's first tenet [4]. The mean value theorem by Lagrange is frequently applied in many areas of mathematics. It has historically been extended in a variety of ways by numerous mathematicians. Direct generalizations of this theorem include that Delle's formula and Cauchy's mean value theorem. Yang had conducted additional research in 1963 [5], and Stieltjes and Scharz had further generalized this theorem. In reality, the mean value theorem of integrals and the mean value theorem of differentials are theorems of the existence of middle points in an interval that is appropriate for a given equation. The middle point can be verified by the mean value theorem, but it does not provide a way to pinpoint its exact location. Li talked about it in 1985 [6]. The integral mean value theorem's tendency to hold when both ends of an interval converge to a fixed point within it was also discovered by the authors in 1994. This finding generalizes the asymptotic findings of the theorem's integral mean value when the interval's endpoints [7]. The study of the asymptotic state of the middle point of the mean value theorem of integrals also emerged after the 20th century [8].

The primary objective of this essay is to demonstrate some significant mean value theorems of differentiation and, from there, to generalize them to other areas, such as mean value theorems of integrals.

2. Differential Mean Value Theorem

Theorem 2.1 ([9]) When the function of meets the requirements listed below:

- (1) On a closed interval $[a_1, a_2]$, g is continuous.
- (2) g can be distinguished on the open interval (a_1, a_2) .
- (3) $g(a_1) = g(a_2)$.

Then, it can find at least one-point ξ satisfying:

$$g'(\xi) = 0. \tag{1}$$

Proof: Since g is a continuous function on a closed interval $[a_1, a_2]$, a maximum and a minimum must exist, which are indicated by M and m , respectively. There are two requirements:

- (1) G must be a constant on $[a_1, a_2]$ if $M=m$. It is clear that the conclusion is accurate.
- (2) If $m > M$, at least one point is in the maximum M and at least one point is in the lowest m at some places on (a_1, a_2) . Since $g(a_1)=g(a_2)$, the extreme point of g is. As a result of the Fermat Theorem and condition (iii), it is evident that:

$$g'(\xi) = 0. \tag{2}$$

Theorem 2.2 (Lagrange Mean Value Theorem) When the function g has the following three factors:

- (1) g is a continuous function on a closed set $[x_1, x_2]$,
- (2) g is a differentiable function on an open set (x_1, x_2) ,
- (3) $g(x_1) = g(x_2)$.

Then, at least one-point ξ satisfies:

$$g'(\xi) = \frac{g(x_2)-g(x_1)}{x_2-x_1}. \tag{3}$$

Proof: Construct an auxiliary function:

$$G(x) = g(x) - g(x_1) - \frac{g(x_2) - g(x_1)}{x_2 - x_1} (x - x_1). \tag{4}$$

It is obvious that $G(x_1) = G(x_2) = 0$ and G satisfies the other two conditions of the Rolle Mean Value Theorem. So, there will be $\xi \in (x_1, x_2)$ such that:

$$G'(\xi) = g'(\xi) - \frac{g(x_2) - g(x_1)}{x_2 - x_1} = 0. \tag{5}$$

This finishes the proof of (3).

Theorem 2.3 (Cauchy Mean Value Theorem) Assume that the functions g satisfies that:

- (1) On a closed interval $[x_1, x_2]$, they are all continuous.
- (2) On an open interval, they are all differentiable (x_1, x_2) .
- (3) $g'(x)$ and $h'(x)$ not equal to 0 at the same time,
- (4) $h(x_1) \neq h(x_2)$.

Then there will be $\xi \in (x_1, x_2)$ satisfying

$$\frac{g'(\xi)}{h'(\xi)} = \frac{g(x_2) - g(x_1)}{h(x_2) - h(x_1)}. \tag{6}$$

Proof: Create an auxiliary function as evidence:

$$G(x) = g(x) - g(x_1) - \frac{g(x_2) - g(x_1)}{h(x_2) - h(x_1)} (h(x) - h(x_1)). \tag{7}$$

There will be (x_1, x_2) because it is apparent that G satisfies the Rolle Mean Value Theorem on $[x_1, x_2]$.

$$G'(\xi) = g'(\xi) - \frac{g(x_2) - g(x_1)}{h(x_2) - h(x_1)} h'(\xi) = 0. \tag{8}$$

It is possible to convert this equation into the form of equation (6).

Example 2.1 Try to prove that $\arctan a_2 - \arctan a_1 < a_2 - a_1$.

Proof: Assume that $g(x) = \arctan x$, then

$$g(a_2) - g(a_1) = g'(\xi)(a_2 - a_1) = \frac{1}{1 + \xi^2} (a_2 - a_1), \quad a_1 < \xi < a_2. \tag{9}$$

So,

$$\arctan a_2 - \arctan a_1 < a_2 - a_1. \tag{10}$$

Example 2.2 Assume that the function g is continued on a closed interval $[x_1, x_2]$ and differentiable on an open interval (x_1, x_2) . Try to prove that there is $\xi \in (x_1, x_2)$ which makes

$$g(x_2) - g(x_1) = \xi \cdot g'(\xi) \cdot \ln \frac{x_1}{x_2}. \tag{11}$$

Proof: Assume $h(x) = \ln x$. It is obvious that it satisfies the four conditions in Theorem 2.3 with $g(x)$ on the closed interval $[x_1, x_2]$. Then it holds that there exists $\xi \in (x_1, x_2)$ which makes

$$\frac{g(x_2) - g(x_1)}{\ln x_2 - \ln x_1} = \frac{g'(\xi)}{\frac{1}{\xi}}. \tag{12}$$

Then it will be the equation (11) after changing the form of it.

3. Application

Theorem 3.1 (Fundamental formula of calculus, [10]) If the function $G(x)$ is a primitive function of a continue function $g(x)$ on the closed interval $[x_1, x_2]$, then

$$\int_{x_1}^{x_2} g(x)dx = G(x_2) - G(x_1). \quad (13)$$

Proof: Insert a number of points in $[x_1, x_2]$ which is a closed interval

$$x_1 = a_0 < a_1 < a_2 < \dots < a_{n-1} < a_n = x_2. \quad (14)$$

So it's obvious that

$$G(x_2) - G(x_1) = \sum_{i=1}^n [G(a_i) - G(a_{i-1})]. \quad (15)$$

For each term on the right side of the function above, the differential mean value theorem follows that

$$G(x_2) - G(x_1) = \sum_{i=1}^n G'(\xi_i)(a_i - a_{i-1}) = \sum_{i=1}^n g(\xi_i)(a_i - a_{i-1}), \quad a_{i-1} < \xi_i < a_i. \quad (16)$$

Assume that $\Delta a_i = a_i - a_{i-1}$, $i=1, 2, 3, \dots, n$, $\lambda = \max\{\Delta a_1, \Delta a_2, \dots, \Delta a_n\}$. According to the definition of a definite integral. The continue function $g(x)$ can be integrate on $[a, b]$, then

$$G(b) - G(a) = \lim_{\lambda \rightarrow 0} \sum_{i=1}^n g(\xi_i)(x_i - x_{i-1}) = \int_a^b g(x)dx. \quad (17)$$

It should be observed that the aforementioned evidence is not only quite straightforward but also, and perhaps more importantly, directly reflects the practical significance. Instead of the normal conclusions drawn from the idea of the derivative of the function, a number of equations are utilized to demonstrate the relationship between the function's increment and the rate of change for the function. For instance, it could be helpful to translate the aforementioned proof into a precise and suitable physical sense. As a basic function of $v(t)$, which is the velocity function, the position function $s(t)$ of a linear motion with variable speed is connected to the velocity function at the time interval $[T_1, T_2]$ by the following equation,

$$s(T_2) - s(T_1) = \sum_{i=1}^n [s(t_i) - s(t_{i-1})] = \sum_{i=1}^n v(\tau_i)(t_i - t_{i-1}), \quad t_{i-1} < \tau_i < t_i, \quad (18)$$

$$s(T_2) - s(T_1) = \lim_{\lambda \rightarrow 0} \sum_{i=1}^n v(\tau_i)(t_i - t_{i-1}) = \int_{T_1}^{T_2} v(t)dt. \quad (19)$$

It is clear from this proof procedure that the relationship between the definite integral of the continuous function $g(x)$ on the interval $[a, b]$ and the increment of the original function of $g(x)$ on the interval $[a, b]$ is equal rather than being as indirect as is typically the case.

Theorem 3.2 (Mean Value Theorem of Integrals) If the function $g(x)$ is a continuous function on the closed set $[x_1, x_2]$, then there will be at least one-point ξ on the open interval (x_1, x_2) which makes

$$\int_{x_1}^{x_2} g(x)dx = g(\xi)(x_2 - x_1), \quad x_1 < \xi < x_2. \quad (20)$$

Proof: The continuous function $g(x)$ possesses the primitive function on a closed interval $[x_1, x_2]$ according to the original function's existence theorem. The Differential Mean Value Theorem and the Basic Calculus Formula show that:

$$\int_{x_1}^{x_2} g(x)dx = G(x_2) - G(x_1) = G'(\xi)(x_2 - x_1) = g(\xi)(x_2 - x_1) \quad (21)$$

$$G(x_2) - G(x_1) = g(\xi)(x_2 - x_1) = \int_{x_1}^{x_2} g(x)dx \quad (22)$$

Finally, it should be mentioned that, with the exception of the infinitely many internal discontinuity spots of the first kind, $g(x)$ is a continuous function on the closed set $[x_1, x_2]$. Theorem 3.1's conclusion is still true in this case. Theorem 3.2's conclusion is invalid in this instance, even if $x_1 = x_2$. On the closed interval $[1, 2]$, the following pointwise continuous function $g(x)$ and its original function $G(x)$ are given.

$$g(x) = \begin{cases} 0, & 0 \leq x \leq 1, \\ 1, & 1 < x \leq 2, \end{cases} \quad G(x) = \begin{cases} C, & 0 \leq x \leq 1, \\ x + C - 1, & 1 < x \leq 2. \end{cases} \quad (23)$$

The conclusion of Theorem 3.1 is valid, just $G(2) - G(0) = 1 = \int_0^2 g(x)dx$. However, the conclusion of Theorem 3.2 is not valid.

4. Conclusion

The mean value theorem is the theoretical cornerstone for calculus. It is a significant theorem that illustrates the link between functions and derivatives. It has numerous applications in formula derivation and theorem proof. It also plays a significant role in a variety of contexts. In this paper, session 2 builds the groundwork for the remaining material and provides explicit proofs for the three basic differential mean value theorems. The main focus of Session 3 is to expand the earlier foundational material and demonstrate the Mean Value Theorem of Integrals and the Calculus foundational Theorem using the Mean Value Theorem of Differentials. The mean value theorem is a key and significant component of mathematical analysis, from which numerous significant and crucial conclusions and proofs can be deduced, making this research significant. In order to contribute to and serve as a reference for future research, this work examines the pertinent extensions of the mean value theorem's issues.

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