

# Cauchy's Integral Formula and Its Applications in Calculating Integrals

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**Abstract.** Cauchy's Integral Theorem is a crucial mathematical theorem that describes the behavior of a complex function on a complex plane. This paper reviews the proof of Cauchy's integral theorem and formula, and the deduction of Cauchy's integral like Cauchy's integral formula with the point in the numerator is on the contour integral. This paper also explores Cauchy's integral when the function contains a non-analytic point inside the contour integral. The way is splitting the entire integral into the sum of several contour integrals equivalently, putting the function  $f(z)$  in the numerator into Laurent Series, then simplifying each contour integral using different methods including the derivative of Cauchy's Integral. The result is the contour integral around point  $z = z_0$  minus part of the Laurent Series of  $f(x)$  in the numerator. This paper shows a path to deal with Cauchy's integral when the function inside the contour integral is not holomorphic in the region.

**Keywords:** Cauchy's integral formula, the derivative of Cauchy's integral formula, non-analytic point.

## 1. Introduction

Complex analysis is a specialized mathematical field that studies functions using complex numbers, revealing their unique traits. It introduces complex numbers, which are crucial for solving non-traditional problems. Topics include analytic, holomorphic, and harmonic functions, along with conformal mappings. A core feature is the Cauchy integral formula, which provides innovative ways to define and evaluate analytic functions. Since Cauchy's integral is really important, a lot of research was conducted. For example, in 1996, Xie developed the integration technique for double analytic functions, leading to the formulation of Morera's theorem, and Cauchy's integral theorem applicable to the functions [1]. In 2009, Bie explored Cauchy's integral formula in superspace [2]. In 2010, Xu and Du explored Cauchy's integral formula, and Cauchy-Pompeiu formula within the context of generalized Clifford analysis [3]. In 2010, Du explored the boundary characteristics of Cauchy-type integrals in the context of hypercomplex analysis [4]. In 2012, Gong explored Cauchy's integral formula in  $R^n$  space [5]. In 2013, Xu explored the generalization of Cauchy's integral and derivation formula [6]. Cauchy's integral formula also plays a crucial role in different fields. In 2011, Zhao researched the application of Cauchy's integral in teaching the Math Physics method [7]. In 2015, Wang researched the vessel segmentation algorithm Based on Octonion Cauchy integral formula and SteiN-Weiss analytical function properties [8-10].

In this essay, the proof of Cauchy's integral formula and its deductions are discussed. Through the integral formula and its deductions, and the Laurent expansion, the Cauchy integral formula with unresolvable points is discussed.

## 2. Cauchy's integral formula

**Theorem 2.1** If  $f(z)$  is a holomorphic function in region D, for every  $z_0$  in Region D,

$$f(z_0) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z-z_0} dz, \quad (1)$$

While  $\gamma$  is a counterclockwise simple connected close curve in region D contains  $z = z_0$ .

In order to establish Cauchy's Integral Formula, our initial task entails demonstrating Cauchy's Integral Theorem. When a function  $f(z)$  is holomorphic within the confines of region D in the complex plane, the integral along any closed and smooth curve  $\gamma$  within region D is equal to zero:

$$\oint_{\gamma} f(z) dz = 0. \tag{2}$$

Proof of equation (2): Since the independent is:  $z = x + iy$ , it is sure that the function  $f(z)$  can be written into:  $f(z) = u(x, y) + iv(x, y) = \mathbf{u} + i\mathbf{v}$ .

In this case, it follows:

$$\begin{aligned} \oint_{\gamma} f(z) dz &= \oint_{\gamma} (\mathbf{u} + i\mathbf{v}) (dx + idy) \\ &= \oint_{\gamma} (\mathbf{u} dx - v(x, y) dy) + i \oint_{\gamma} (\mathbf{u} dy + v(x, y) dx). \end{aligned} \tag{3}$$

According to Green's theory, both contour integrals with a double integral in region D can be replaced:

$$\oint_{\gamma} (-v dy + \mathbf{u} dx) = \iint_D \left( -\frac{\partial v}{\partial x} - \frac{\partial \mathbf{u}}{\partial y} \right) dx dy. \tag{4}$$

$$\oint_{\gamma} (v dx + \mathbf{u} dy) = \iint_D \left( -\frac{\partial v}{\partial y} + \frac{\partial \mathbf{u}}{\partial x} \right) dx dy. \tag{5}$$

Since the function  $f(z)$  is holomorphic in region D, which means that it satisfies the Cauchy–Riemann equations which is:

$$\frac{\partial v}{\partial y} = \frac{\partial \mathbf{u}}{\partial x}. \tag{6}$$

$$-\frac{\partial v}{\partial x} = \frac{\partial \mathbf{u}}{\partial y}. \tag{7}$$

Therefore, both contour integrals are equal to zero which means that  $\oint_{\gamma} f(z) dz = 0$ .

**Proof:** A contour integral is constructed as shown in the Fig 1:  $\gamma_3$  is an arc with center a with radius r. The m, p are points close to each other. The n, q are points close to each other, segment MN and PQ are parallel. Define that:

$$g(z) = \frac{f(z)}{z - z_0}. \tag{8}$$

Because the function is holomorphic within the curve, according to Cauchy's integral theorem:

$$\oint_{\gamma} g(z) dz = 0. \tag{9}$$

Splitting the contour integral into pieces, the case is that:

$$\sum_{j=1}^4 \int_{\gamma_j} g(z) dz = 0. \tag{10}$$

As points m, p sticks together, and points n, q sticks together,  $\gamma_2$  and  $\gamma_4$  are in same path and opposite direction:

$$\int_{\gamma_2} g(z) dz + \int_{\gamma_4} g(z) dz = 0. \tag{11}$$

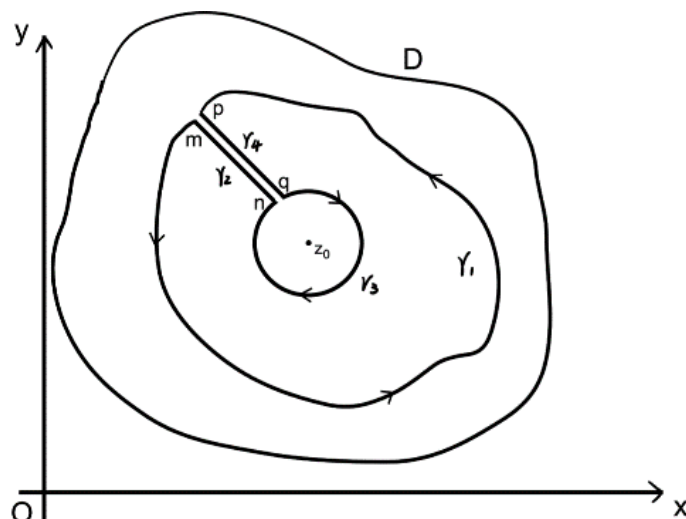


Figure 1. Contour of  $\gamma_3$

So, it follows that that:

$$\oint_{\gamma_1} g(z) dz = \oint_{\gamma_3^-} g(z) dz. \tag{12}$$

Since points m, p and points n, q sticks together,  $\gamma_1$  and  $\gamma_3$  are both contour integral. Since,  $\gamma_3$  is an arc with center a with radius r,  $\gamma_3$  can be replaced with:

$$\gamma_3^- = z_0 + re^{i\theta}, \theta \in [0,2\pi). \tag{13}$$

By replacing  $\gamma_3$  with the equation, the case becomes:

$$\oint_{\gamma_1} g(z) dz = \oint_{\gamma_3^-} g(z) dz = \int_0^{2\pi} \frac{f(z_0+re^{i\theta})}{re^{i\theta}} ire^{i\theta} dz = 2\pi i \times f(z_0 + re^{i\theta}). \tag{14}$$

Since  $f(z)$  is continuous, as r approach to 0, and the result will be:

$$\lim_{r \rightarrow 0} 2\pi i \times f(z_0 + re^{i\theta}) = 2\pi i \times f(z_0). \tag{15}$$

Therefore,

$$f(z_0) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z-z_0} dz. \tag{16}$$

### 2.1. Derivative of Theorem 2.1 and its application

**Theorem 2.2** If  $f(z)$  is a holomorphic function in region D, for every  $z_0$  Region D,

$$f^n(z_0) = \frac{n!}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z-z_0)^{n+1}} dz, n \in 0 \cup Z^+, \tag{17}$$

While  $\gamma$  is a counterclockwise simple connected close curve in region D which contains  $z = z_0$ . This theorem can be proved by math induction:

When  $n = 0$ , the equation is true obviously. Assume that when  $n = k$  the statement is true:

$$f^k(z_0) = \frac{k!}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z-z_0)^{k+1}} dz. \tag{18}$$

$f^{(k+1)}(z_0)$  will be:

$$\begin{aligned}
 f^{k+1}(z_0) &= \frac{f^k(z_0+\Delta z) - f^k(z_0)}{\Delta z} \\
 &= \frac{1}{\Delta z} \frac{k!}{2\pi i} \oint_{\gamma} f(z) \left( \frac{1}{(z-(z_0+\Delta z))^{k+1}} - \frac{1}{(z-z_0)^{k+1}} \right) dz \\
 &= \frac{1}{\Delta z} \frac{k!}{2\pi i} \oint_{\gamma} f(z) \left( \frac{(z-z_0)^{k+1} - \sum_{m=0}^{k+1} \binom{k+1}{m} (-1)^m (\Delta z)^m (z-z_0)^{k+1-m}}{(z-z_0)(z-(z_0+\Delta z))^{k+1}} \right) dz \\
 &= \frac{k!}{2\pi i} \oint_{\gamma} f(z) \left( \frac{(k+1)(z-z_0)^k - \sum_{m=2}^{k+1} \binom{k+1}{m} (-1)^m (\Delta z)^{m-1} (z-z_0)^{k+1-m}}{(z-z_0)(z-(z_0+\Delta z))^{k+1}} \right) dz.
 \end{aligned} \tag{19}$$

Let:

$$\begin{aligned}
 O(\Delta z) &= - \sum_{m=2}^{k+1} \binom{k+1}{m} (-1)^m (\Delta z)^{m-1} (z-z_0)^{k+1-m} \\
 &= \frac{k!}{2\pi i} \oint_{\gamma} f(z) \left( \frac{(k+1)(z-z_0)^k + O(\Delta z)}{(z-z_0)(z-(z_0+\Delta z))^{k+1}} \right) dz \\
 &= \frac{k!}{2\pi i} \oint_{\gamma} \left( (k+1) \frac{f(z)}{(z-z_0)^{k+2}} \left( \frac{z-z_0}{z-z_0-\Delta z} \right)^{k+1} + \frac{f(z) \times O(\Delta z)}{(z-z_0)(z-(z_0+\Delta z))^{k+1}} \right) dz
 \end{aligned} \tag{20}$$

As  $\Delta z$  approaches to 0, the equation becomes:

$$f^{k+1}(z_0) = \frac{(k+1)!}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z-z_0)^{k+2}} dz. \tag{21}$$

Since when  $n = 0$ , the statement is correct. If formula is true when  $n = k$ , formula is also correct when  $n = k + 1$ . So, the case for even non-negative  $n$  can be proved:

$$f^n(z_0) = \frac{n!}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z-z_0)^{n+1}} dz. \tag{22}$$

## 2.2. Application of Theorem 2.2

Obviously, the function  $f(z) = \frac{1}{z-3}$  is holomorphic inside of the curve  $\gamma = |z|$ , and  $z = i$  is the only singular point inside the contour integral  $\gamma = |z|$ . **Theorem 2.1** leads that:

$$\oint_{\gamma=|z|} \frac{1}{(z-3)(z-i)^5} dz = \frac{2\pi i}{4!} \frac{d^4}{dz^4} \frac{1}{z-3} \Big|_{z=i} = 2\pi i \frac{1}{(i-3)^5}. \tag{23}$$

**Theorem 2.3** If  $f(z)$  is a holomorphic function in region  $D$ , for every  $z_0$  on curve  $\gamma$ ,

$$f(z_0) = \frac{1}{\pi i} \oint_{\gamma} \frac{f(z)}{z-z_0} dz, \tag{24}$$

While  $\gamma$  is a counterclockwise simple connected close curve in region  $D$ .

Proof: As shown in Fig. 2,  $z_0$  is the center of the circle  $\gamma_2 + \gamma_3$ , which radius is  $r$ .

Define that

$$g(z) = \frac{f(z)}{z-z_0}. \tag{25}$$

It is quite obvious that:

$$\begin{aligned} \oint_{\gamma_1} g(z) dz &= \oint_{\gamma_2} g(z) dz = \oint_{\gamma_3} g(z) dz = \oint_{\gamma_4} g(z) dz \\ \oint_{\gamma_1} g(z) dz &= \frac{1}{2} \left( \oint_{\gamma_2} g(z) dz + \oint_{\gamma_4} g(z) dz \right) \\ &= \frac{1}{2} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} ire^{i\theta} dz = \pi i \times f(z_0 + re^{i\theta}). \end{aligned} \tag{26}$$

Because  $f(z)$  is a continuous function, as  $r$  approaches to 0, the result will be:

$$\lim_{r \rightarrow 0} \pi i f(z_0 + re^{i\theta}) = \pi i f(z_0). \tag{27}$$

So, it follows that:

$$f(z_0) = \frac{1}{\pi i} \oint_{\gamma} \frac{f(z)}{z-z_0} dz, z_0 \in C. \tag{28}$$

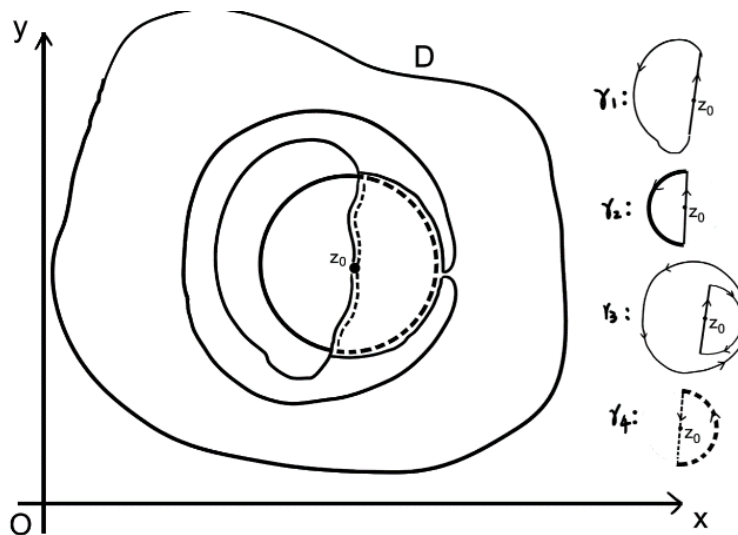


Figure 2. Contour of  $\gamma$

### 2.3. Non-analytic point $z_1$ inside the contour integral

**Theorem 2.4** If  $f(z)$  is a holomorphic function except for  $z = z_1$  in region  $D$ . Laurent Series of  $f(z)$  at  $z = z_1$ :  $f(z) = \sum_{m=0}^{\infty} a_m(z - z_1)^m + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_1)^n}$ . Then

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z-z_0} dz = \begin{cases} f(z) - \sum_{n=1}^{\infty} \frac{b_n}{(z-z_1)^n}, & z_0 \text{ is inside of } \gamma \\ \frac{1}{2} f(z) - \sum_{n=1}^{\infty} \frac{b_n}{(z-z_1)^n}, & z_0 \text{ is on } \gamma \\ - \sum_{n=1}^{\infty} \frac{b_n}{(z-z_1)^n}, & z_0 \text{ is outside of } \gamma \end{cases} \tag{29}$$

Proof: As shown in Fig.3, it is quite obvious that

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z-z_0} dz = \frac{1}{2\pi i} \oint_{\gamma_0} \frac{f(z)}{z-z_0} dz + \frac{1}{2\pi i} \oint_{\gamma_1} \frac{f(z)}{z-z_0} dz. \tag{30}$$

According to the text above, it is known that:

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z-z_0} dz = \begin{cases} f(z), z_0 \text{ is inside of } \gamma \\ \frac{1}{2} f(z), z_0 \text{ is on } \gamma \\ 0, z_0 \text{ is outside of } \gamma \end{cases} \quad (31)$$

The second part is:

$$\frac{1}{2\pi i} \oint_{\gamma_1} \frac{f(z)}{z-z_0} dz = \frac{1}{2\pi i} \oint_{\gamma_1} \frac{\sum_{m=0}^{\infty} a_m(z-z_1)^m}{z-z_0} dz + \frac{1}{2\pi i} \oint_{\gamma_1} \frac{\sum_{n=1}^{\infty} \frac{b_n}{(z-z_1)^n}}{z-z_0} dz. \quad (32)$$

The first part of the equation is 0 because the function  $\frac{\sum_{m=0}^{\infty} a_m(z-z_1)^m}{z-z_0}$  is analytic inside  $\gamma_1$ . According to the **Theorem 2.2**, the second part of the equation is:

$$\begin{aligned} \frac{1}{2\pi i} \oint_{\gamma_1} \frac{\sum_{n=1}^{\infty} \frac{b_n}{(z-z_1)^n}}{z-z_0} dz &= \sum_{n=1}^{\infty} \frac{b_n}{2\pi i} \oint_{\gamma_1} \frac{1}{(z-z_1)^n} dz \\ &= \sum_{n=1}^{\infty} \frac{b_n}{(n-1)!} \frac{d^{(n-1)} \frac{1}{z-z_0}}{dz^{(n-1)}} \Big|_{z=z_1} \\ &= - \sum_{n=1}^{\infty} \frac{b_n}{(z_0-z_1)^n}. \end{aligned} \quad (33)$$

So, the prove is done.

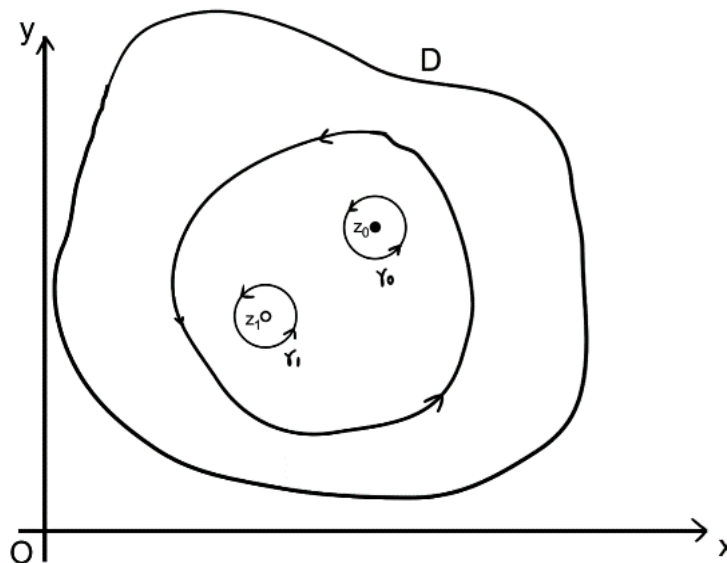


Figure 3. Contour of  $\gamma$  in Theorem 2.3.2

### 3. Conclusion

In this essay, Cauchy's integral formula and its expansion are explored. The proof of Cauchy's integral formula is reviewed. Some deductions of Cauchy's integral is also proved. Based on the Cauchy's integral and its deduction, the essay explores Cauchy's integral formula when the function contains a non-analytic point inside the contour integral. The result is obtained by putting the function numerator into Laurent Series and then simplifying the integral using the derivative of the Cauchy's integral formula. As last, a concise answer, which is related to part of the Laurent expansion of the function in the numerator, is obtained. On one hand, the proof of Cauchy's integral formula and its deduction are reviewed. On the other hand, it provides a path to calculate Cauchy's integral when the function is non-holomorphic.

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