Applications of Cauchy-Riemann Equations on Several Examples

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Abstract. Cauchy-Riemann equations is always a heated topic in mathematics. It is widely used to solve both mathematic problems and daily issues. In this paper, basic knowledge which is needed for the study of Cauchy-Riemann equations has been explained carefully. This includes the understanding of analytic function, derivatives, and partial derivatives, as well as the method to derive the final equations. The method mainly used to obtain the equations in the paper is to calculate the limits of a function with complex variables along two axes. The application that Cauchy-Riemann equations are most frequently applied to is to find derivatives of complex functions, and some relevant examples are included. In addition, there is an extension of the basic use of Cauchy-Riemann equations in functions that are in Cartesian forms. This study can help to inspire the green hands in the field of complex analysis and let them have a good command of knowledge of the most superficial level of this field.

Keywords: Analytic function, Derivatives, Partial derivatives, Cauchy-Riemann equations.

1. Introduction

The Cauchy Riemann equation, which is named in honor of Augustin Cauchy and Bernhard Riemann, is well known in the world of mathematics, especially in complex analysis. It is generally a useful tool to test whether at some point, a complex function can be differentiated. The Cauchy Riemann equation was first used by Jean Le Rond. After that, Leonhard Euler found that the equation was related to the analytic function. The Cauchy-Riemann equation is the result of a bunch of mathematical symbols such as definitions, theorems, inferences, and examples under the traditional packaging of Euclidean deduction [1].

Cauchy-Riemann equations are so useful that they can be used in various ways. For example, Cauchy-Riemann equations can be adapted in a method of image recovery [2], which is based on the use of noniterative phase retrieval. The Cauchy-Riemann equations are applied to assess phase derivatives using magnitude derivatives. Cauchy-Riemann equations can also be related to deeper topic like new extension phenomenon called sectorial analyticity property discovered by Kossovskiy [3]. They found a phenomenon of analytical continuation for Cauchy-Riemann diffeomorphisms, and it seems to be the best regularity characteristic for Cauchy-Riemann diffeomorphisms generally. In the field of geometry, the Schwarz–Poisson formula is derived with the help of Cauchy-Riemann equations together with the plane parqueting technique according to Wang [4], who has studied the nonhomogeneous Schwaez-type boundary-value problem on an isosceles orthogonal triangle. Cauchy-Riemann equations are suggested by Gimeno to convert Green's functions determined in Euclidean space into Green's functions in non-perturbative quantum field theory in Minkowski space [5]. The extension of Cauchy-Riemann equations is always being investigated by people. In the studies of AB Rasulov [6], the general solution of a generalized Cauchy-Riemann system whose coefficients admit higher-order singularities on a segment is worked out as an integral representation.

In this paper, knowledges that are related to Cauchy-Riemann equations are explained in detail in the section 2. It included analytic function, derivatives and partial derivatives, the derivation process of the Cauchy-Riemann equations, and the way of applying in different forms. The section 3 shows some examples of the application of concepts mentioned above. In the beginning, there are two examples of calculating the partial derivatives. The next three examples involve direct use of Cauchy-

Riemann equations to test whether a function has derivatives or not. The last part of this section are two examples about polar coordinate. The section 4 is a conclusion of the whole paper.

2. Methods

2.1. Analytic Functions

A definition of analytic function according to Saks is that every (non-empty) family D of analytic elements with centers in a region G, and it should satisfy two conditions [7]. The first condition is that given any two elements of the family D, one is a continuation of the other in the region G, and the second one is that every analytic element which is a continuation in the region G of an analytic element belonging to D belongs to D, will be called analytic function. If G is the entire plane, an analytic function in G will be called simply analytic function. A simpler way of defining it is that a function $f(z)$ in the domain D is analytical (D is included by the complex plane) if $f(z)$ is single valued and has a finite derivative $f'(z)$ for each z, where z is in the domain D. For a complex function to be analytic, the derivatives must be the same for the approach from different directions.

In order to better understand the nature of an analytic function, an example of what kind of function is not analytic is shown. The function is $f(z) = \bar{z} = x - iy$, then one computes its derivative,

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\bar{z}}{z} \tag{1}$$

Let $\Delta z = \Delta x + i\Delta y$; $\overline{\Delta \mathrm{z}} = \Delta x - i\Delta y$, and then the equation becomes

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{\Delta x + i\Delta y}{\Delta x - i\Delta y}. \tag{2}$$

When approaching this derivative as Δz goes to zero in different directions, something unexpected will happen. In the first case, starting from the real axis and approach the origin as Δx goes to zero. And let Δy to be zero. ($\Delta y = 0$). At this moment, the expression becomes,

$$\frac{df}{dz} = \lim_{\Delta x \to 0} \frac{\Delta x + 0}{\Delta x + 0} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = +1 \tag{3}$$

In the second case, starting from the imaginary axis and approach the origin as Δy goes to zero. This time let Δx to be zero. ($\Delta x = 0$). Similarly, the expression then becomes,

$$\frac{df}{dz} = \lim_{\Delta y \to 0} \frac{0 - \Delta y i}{0 + \Delta y i} = \lim_{\Delta y \to 0} \frac{-\Delta y i}{\Delta y i} = -1 \tag{4}$$

The outcome is not the same as Δz approaching the origin in different directions. That means the function, the complex conjugate of z does not have a well-defined derivative. In another words, it does not have a derivative that is independent of the path of approach the Δz goes to zero. The followings are some examples of analytic functions. There are generally two categories. The first one is polynomial: z^n for integer n. The second one is functions with a Taylor series, which include e^z, $\sin z$ and $\cos z$.

2.2. Derivatives and Partial Derivatives

According to Brown and Churchill [8], the derivative at z_0 is the limit,

$$f'(z_0) = \lim_{z \to 0} \frac{f(z) - f(z_0)}{z - z_0}. \tag{5}$$

In addition, the function is said to be differentiable when $f'(z0)$ exists. Next, express the variable z in a different form $\Delta z = z - \mathrm{z}_0$. The equation turns out to be,

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} \tag{6}$$

Let say there is a function that depends on both x and y, and it is called $f(x, y)$. Unlike the normal functions, it will end up being a surface in space instead of a curve. In this way, how to find the rate of change for that surface? In order to do that, it is necessary to find the rate of change in specific directions independently.

If starting with computing the speed at which x's direction is changing, it just means to differentiate with respect to x, and treat y as a constant. There is a new notation for this: $\frac{\delta}{\delta x}$. Similarly, if computing the speed at which y's direction is changing, this turns out to be a partial derivative with respect to y. Therefore, x will be treated as a constant.

2.3. Cauchy-Riemann Equations

Similar as what has been mentioned before, in order to let a function to be analytic, the derivative of it must be same for the two paths on real and imagery axis [9], Let $f(z) = u(x, y) + iv(x, y)$ and $z = x + iy$, then,

$$f(z + \Delta z) - f(z) = [u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y)] - [u(x, y) + iv(x, y)], \tag{7}$$

And

$$\frac{df}{dz} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x+\Delta x,y+\Delta y)-u(x,y)}{\Delta x+i\Delta y} + i\,\frac{v(x+\Delta x,y+\Delta y)-v(x,y)}{\Delta x+i\Delta y}. \tag{8}$$

In the first case, approaching on the real axis, which means Δy starts out being equal to zero. $(\Delta y = 0)$ The limit then becomes,

$$\frac{df}{dz} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x,y)-u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x,y)-v(x,y)}{\Delta x} = \frac{\delta u}{\delta x} + i\,\frac{\delta v}{\delta x}. \tag{9}$$

In the second case, approaching on the imagery axis, which means this time $\Delta x = 0$. The limit then becomes,

$$\frac{df}{dz} = \lim_{\Delta y \to 0} \frac{u(x,y+\Delta y)-u(x,y)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x,y+\Delta y)-v(x,y)}{i\Delta y}. \tag{10}$$

Because $\frac{1}{i} = -i$, then

$$\frac{df}{dz} = \lim_{\Delta y \to 0} \frac{v(x,y+\Delta y)-v(x,y)}{\Delta y} - i \lim_{\Delta y \to 0} \frac{u(x,y+\Delta y)-u(x,y)}{\Delta y} = \frac{\delta v}{\delta y} - i\,\frac{\delta u}{\delta y}. \tag{11}$$

In the two cases, the real parts and imagery parts have to be equal respectively,

$$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y} \; and \; \frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}. \tag{12}$$

The two equations are just Cauchy-Riemann equations [10].

2.4. Polar Coordinates

By doing the coordinate transformation for the complex function $f(z) = x + iy$, it is easy to get $x = r\cos\theta$, $y = r\sin\theta$. When there is a $w = f(z)$, the real and imagery part u and v are expressed in the form of either x, y or r, θ. By applying the chain rule for differentiating the function of the two variables, it becomes,

$$\frac{\delta u}{\delta r} = \frac{\delta u}{\delta x} \times \frac{\delta x}{\delta r} + \frac{\delta u}{\delta y} \times \frac{\delta y}{\delta r}, \ \frac{\delta u}{\delta \theta} = \frac{\delta u}{\delta x} \times \frac{\delta x}{\delta \theta} + \frac{\delta u}{\delta y} \times \frac{\delta y}{\delta \theta} \tag{13}$$

Then, the equations for u can be rewritten as,

$$\frac{\delta u}{\delta r} = \frac{\delta u}{\delta x} cos\,\theta + \frac{\delta u}{\delta y} sin\,\theta, \ \ \frac{\delta u}{\delta \theta} = -\frac{\delta u}{\delta x} r\,sin\,\theta + \frac{\delta u}{\delta y} r\,cos\,\theta \tag{14}$$

Similarly, equations for v can be got in the same way,

$$\frac{\delta v}{\delta r} = \frac{\delta v}{\delta x} cos\,\theta + \frac{\delta v}{\delta y} sin\,\theta, \ \ \frac{\delta v}{\delta \theta} = -\frac{\delta v}{\delta x} r\,sin\,\theta + \frac{\delta v}{\delta y} r\,cos\,\theta \tag{15}$$

According to Cauchy-Riemann equations, the equations (3) become,

$$\frac{\delta v}{\delta r} = -\frac{\delta u}{\delta y} cos\,\theta + \frac{\delta u}{\delta x} sin\,\theta, \ \ \frac{\delta v}{\delta \theta} = \frac{\delta u}{\delta y} r\,sin\,\theta + \frac{\delta u}{\delta x} r\,cos\,\theta \tag{16}$$

From Eqs. (2) and (4), it is inferred that

$$r\frac{\delta u}{\delta r} = \frac{\delta v}{\delta \theta}, \frac{\delta u}{\delta \theta} = -r\frac{\delta v}{\delta r}. \tag{17}$$

3. Applications

3.1. Calculating the Partial Derivatives

There are some examples of calculating the partial derivatives. The first example is $f(x, y) = xy^2$. According to the definition of partial derivative, it is easy to get the partial derivative in respect to x $\frac{\delta f}{\delta x} = y^2 + 3x^2$. In addition, partial derivative with regard to y is $\frac{\delta f}{\delta y} = 2xy$. Next example is $f(x, y, z) = \frac{x^5 e^{2z}}{y}$. Similarly, the partial derivatives in respect to x, y and z can be calculated one by one. For x, the partial derivative is $\frac{\delta f}{\delta x} = \frac{5x^4 e^{2z}}{y}$. While for y and z, it is found that $\frac{\delta f}{\delta y} = -\frac{x^5 e^{2z}}{y^2}$ and $\frac{\delta f}{\delta z} = \frac{2x^5 e^{2z}}{y}$.

3.2. Demonstrating the Existence of Derivatives

The first example in this section is $f(z) = 2x + ixy^2$. Firstly, the real part and the imagery part of the function are $u(x, y) = 2x$ and $v(x, y) = xy^2$. Then, the partial derivatives are calculated as $\frac{\delta u}{\delta x} = 2$, $\frac{\delta u}{\delta y} = 0$, $\frac{\delta v}{\delta x} = y^2$, $\frac{\delta v}{\delta y} = 2xy$ respectively. By substituting them in the Cauchy-Riemann equations shown in Eq. (6), it is found that $\frac{\delta u}{\delta x} \neq \frac{\delta v}{\delta y}$ and $\frac{\delta u}{\delta y} \neq -\frac{\delta v}{\delta x}$, indicating that the derivative of $f(z)$ does not exist.

Here is the second example $f(z) = e^x e^{-iy}$. Firstly, rewrite the function as the following,

$$f(z) = e^x e^{-iy} = e^x(cos\,y - isiny) = e^x\,cos\,y - ie^x\,sin\,y. \tag{18}$$

And find the real and imagery components, $u(x,y) = e^x \cos y$ and $v(x,y) = -e^x \sin y$. Calculate the partial derivatives, $\frac{\delta u}{\delta x} = e^x \cos y$, $\frac{\delta u}{\delta y} = -e^x \sin y$, $\frac{\delta v}{\delta x} = -e^x \sin y$, $\frac{\delta v}{\delta y} = -e^x \cos y$.Then, substitute them in the C-R equations, $\frac{\delta u}{\delta x} \neq \frac{\delta v}{\delta y}$ when $y \neq 0$, $\frac{\delta u}{\delta y} \neq -\frac{\delta v}{\delta x}$ when $y \neq 0$. Therefore, f(z) doesn't have a derivative.

The last example of this type of question is $\mathrm{f(z)} = \mathrm{z} - \bar{z}$, which means that $f(z) = x + iy - (x - iy) = 2iy$. Clearly, the real part and the imagery part are $u(x,y) = 0$ and $v(x,y) = 2y$. By using of the basic relations oft he partial derivatives, it is found that

$$\frac{\delta u}{\delta x} = 0, \quad \frac{\delta u}{\delta y} = 0, \quad \frac{\delta v}{\delta x} = 0, \qquad \frac{\delta v}{\delta y} = 2. \tag{19}$$

By substituting them in the Cauchy-Riemann equations $\frac{\delta u}{\delta x} \neq \frac{\delta v}{\delta y}$ and $\frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$. Therefore, f(z) does not have any derivative.

3.3. Applying in Polar Coordinates

The example is $f(z) = 1/z^2$. The function can be written into the exponential form and then into polar form,

$$f(z) = \frac{1}{(re^{i\theta})^2} = \frac{1}{r^2} e^{-i2\theta} = \frac{1}{r^2}(cos\, 2\theta - i\, sin\, 2\theta). \tag{20}$$

It is obvious that the real and imagery components are $u = \frac{\cos 2\theta}{r^2}$ and $v = -\frac{\sin 2\theta}{r^2}$. Then by calculating the partial derivatives and substitute them in the Eq. (17), it is found that

$$r\frac{\delta u}{\delta r} = -\frac{2\, cos\, 2\theta}{r^2} = \frac{\delta v}{\delta \theta}, \frac{\delta u}{\delta \theta} = -\frac{2\, sin\, 2\theta}{r^2} = -r\frac{\delta v}{\delta r}. \tag{21}$$

Therefore, the derivative of $f(z)$ exists.

The next example is $f(z) = \sqrt{r} e^{i\theta/2}$ $(r > 0, \alpha < \theta < \alpha + 2\pi)$. By rewriting it, the two parts of the complex function can be found as $u = \sqrt{r} \cos\frac{\theta}{2}$ and $v = \sqrt{r} \sin\frac{\theta}{2}$. The relations between the partial derivatives are

$$r\frac{\delta u}{\delta r} = \frac{\sqrt{r}}{2}\cos\frac{\theta}{2} = \frac{\delta v}{\delta \theta}, \ \frac{\delta u}{\delta \theta} = -\frac{\sqrt{r}}{2}\sin\frac{\theta}{2} = -r\frac{\delta v}{\delta r}. \tag{22}$$

Since the Cauchy-Riemann equations are satisfies, the derivative of the function can be found.

4. Conclusion

In conclusion, in this paper, with the help of basic knowledges such as the definition of analytic functions and the method to find limits, Cauchy-Riemann equations are derived from computing derivatives in the vertical and horizontal directions, which in another word, is finding the partial derivatives. After obtaining the equations, the problem of whether a complex function has derivative can be solved easily. In addition, the application of the Cauchy-Riemann equations can be extended from the general Cartesian forms to the polar forms. Since everything is explained in a clear and direct way, this paper helps people, especially high school students who want to have a first insight into complex analysis, to make a good start. By carefully reading through the paper, beginners in mathematics can master the method of computing the partial derivatives, which is an essential skill for the future learning of Math. Besides, it is useful to those who have little knowledge about concepts of complex numbers. In fact, complex numbers are as abstract as people may think, instead, they're just the natural extension of the usual number system from one dimension to two. However, this study of Cauchy-Riemann equations just scratches the surface of the world of complex analysis. As mentioned in the introduction, it can be used in many other fields to help solve some difficult problems rather than just testing the existence of derivatives. And in fact, deeper knowledge behind what have been mentioned in section two such as analytic function and partial derivatives are quite worth investigating as well. Finally, hope that future scholars of complex analysis can get some inspirations from the paper.

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