Real and Complex Analysis of Common Probability Density Functions

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Abstract. In this paper, the most common method of probability density function integration Gaussian Integration is analyzed in real and complex methods. This paper begins by clarifying the meaning and significance of integrating the probability density function. After this, it elucidates and derive the mathematical methods for the analysis of integrals, namely the multiple integral method in real analysis and the residue theorem method in complex analysis. At the same time, the paper also makes it clear to analyze the most typical Gaussian integral in the probability density function. Since then, this integration method has been used respectively for real analysis and complex analysis. Different means of analysis including Pinch criterion or Residue theorem have been applied comprehensively. In this session, different forms and formats of Gaussian integrals are explored to a certain extent. Finally, the benefits of complex analysis and real analysis for the synthesis of probability density function including Gaussian integral are summarized, and the possible applications and prospects of this new analysis method are prospected. However, the limitations of the current approach have also been mentioned.

Keywords: Analytic functions, Complex variables, Probability density functions, Gaussian integral, Residue theorem.

1. Introduction

Probability density function is an important concept in probability theory and statistics, which plays a key role in describing and analyzing the distribution characteristics of random variables [1]. Probability density function has a critical significance in probability theory and statistics. It can not only describe the distribution characteristics of random variables, but also be used in various fields such as statistical inference, stochastic model building, risk assessment and data analysis. Through the study and application of probability density function, the researchers can better understand and analyze random events and random processes, and provide scientific basis for decision making and prediction.

The point of integrating the probability density function is to calculate the probability that a random variable falls within a certain interval. The result of the integral can be understood as the probability of random events occurring in the interval [2]. Specifically, the result of integrating the probability density Function is a cumulative distribution function, which represents the probability that a random variable is less than or equal to a particular value [3]. By integrating the probability density function, the probability distribution of random variables in different intervals can be obtained, and then the probability of random events can be quantified and analyzed.

Gaussian integral is the most common methods for calculating probability density functions, and therefore play an important role in statistics. Gaussian integrals apply to continuous probability distributions, such as normal distributions [4]. Real analysis has always been more common in the calculation of probability density function. Moreover, in recent years, the extension of the latter to the complex plane has attracted more and more academic interest. The purpose of this thesis is to study the real analysis and complex analysis of this most common integraion as canonical example for probability density functions and explore their properties and applications.

2. Integration Method

The purpose of this chapter is to introduce and simply prove two different integration methods, namely multiple integration method and Causelius Residue theorem method. These two methods can effectively solve the real analysis and complex analysis of Poisson integral and Gauss integral respectively, and provide a comprehensive basis for the final conclusion.

2.1. The multiple Integration Method

The double integral of $f(x,y)$ over R has a general algebraic expression [5]

$$\iint_R f(\mathrm{x},\mathrm{y})dA = \lim_{\substack{\Delta x\to\infty \\ \Delta y\to\infty}} \sum f(x_i,y_i)\Delta A \tag{1}$$

For each function of two variables $u(x,y) = f(x)\cdot f(y)$, If the upper and lower limits of the integral are constant, the double integral can be divided into two definite integrals [6], which are

$$I = \iint_D f(\mathrm{x},\mathrm{y})dxdy = \int_a^b dx \int_c^d u(x,y)dy = \int_a^b dx \int_c^d f(x)\mathrm{g}(\mathrm{y})dy \tag{2}$$

And

$$I = \int_a^b f(x)dx \int_c^d \mathrm{g}(\mathrm{y})dy = \int_a^b f(x)dx \cdot \int_c^d g(y)dy, \tag{3}$$

And vice versa.

2.2. Residue Theorem Method

The residue theorem is a very classical rule for integrating functions of complex variables. If a function f resolves everywhere except for several finite singularities inside a simple closed perimeter C, then these singularities must be isolated singularities, see Fig. 1 [7]. On this basis, the residue theorem is a very general description of a clear and comprehensive fact: if f is analytic on C and C is positively oriented, the integral value of f around C is twice the sum of the residues of f at the singularities inside C. If express this simple theorem in a single formula, it is found that

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z). \tag{4}$$

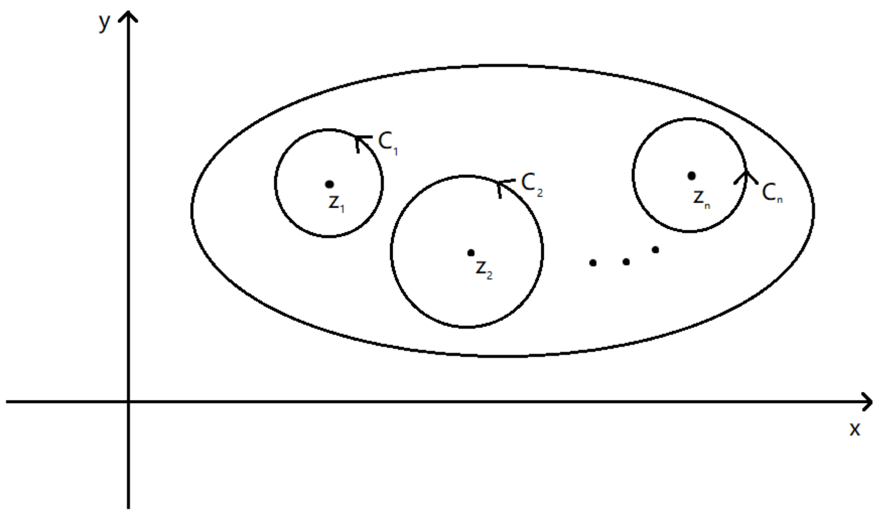

Figure 1. Residue Theorem Illustration of contour in complex plane.

To prove this theorem, it can be assumed that z_k $(k = 1,2, \ldots, n)$ is the center of the positively oriented circles C_k, which fall inside C and are sufficiently small that any two of them have no common points. Obviously, the circumference C_k and the simple closed perimeter C together form the boundary of a closed domain on which f is analytic everywhere, and the interior of which is a multi-connected region of points located inside C and outside each C_k. If Cauchy-Goursat theorem should be applied to such a region at this time, it is obtained that

$$\int_C f(z)dz - \sum_{k=1}^{n} \int_{C_k} f(z)dz = 0. \tag{5}$$

In light of Eq. (4), it is found that

$$\int_{C_k} f(z)dz = 2\pi i \operatorname*{Res}_{z=z_k} f(z). \tag{6}$$

Where $k = 1,2, \ldots n$. Now the proof has been completed.

3. Analysis of Gaussian Integrals

The main content of this part is calculating the general Gaussian integral

$$I = \int_{-\infty}^{+\infty} e^{-x^2} dx \tag{7}$$

In Real analysis and complex integral.

3.1. Real Analysis

This section focuses on the calculation of two different forms of Gaussian integrals using two slightly different methods (Normal multiple integration and Pinch criterion multiple integration) of real analysis.

3.1.1. Normal multiple integration

The gaussian integral can be calculated by the multiple integration method directly [8]. Let

$$I = \int_{-\infty}^{+\infty} e^{-x^2} dx, \tag{8}$$

And it is found that

$$I^2 = \int_{-\infty}^{+\infty} e^{-x^2} dx \cdot \int_{-\infty}^{+\infty} e^{-y^2} dy = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} dxdy. \tag{9}$$

Let $x = r\cos\theta$, $y = r\sin\theta$ ($0 \le \theta \le 2\pi$, $0 \le r < +\infty$), then

$$I^2 = \int_0^{2\pi} d\theta \cdot \int_0^{+\infty} e^{-\left(r^2 cos^2\theta + r^2 sin^2\theta\right)} \cdot r\, dr = \int_0^{2\pi} d\theta \cdot \int_0^{+\infty} e^{-r^2} \cdot r\, dr. \tag{10}$$

Consider the second part of the formula first, it is observed that

$$\frac{1}{2}\int_0^{+\infty} e^{-r^2} dr^2 = -\frac{1}{2} \cdot \int_0^{+\infty} e^{-r^2} d(-r^2) = [-\frac{1}{2} e^{-r^2}]_0^{+\infty} = -\frac{1}{2}(0-1) = \frac{1}{2}. \tag{11}$$

Hence, $I^2 = \int_0^{2\pi} \frac{1}{2} d\theta = \pi$ and $I = \sqrt{\pi}$.

3.1.2. Pinch criterion multiple integration

The gaussian integral can also be computed by multiple integration method with the pinch criterion, as shown in Fig. 2. Let

$$I=\int_0^{+\infty} e^{-x^2}dx=\int_0^{+\infty} e^{-y^2}dy, \tag{12}$$

Which is also a very common form for the Gaussian integral with the lower case zero instead of negative infinity. So,

$$I^2=\int_0^{+\infty} e^{-x^2}dx\cdot\int_0^{+\infty} e^{-y^2}dy. \tag{13}$$

Hence,

$$I^2=\lim_{R\to+\infty}\int_0^R e^{-x^2}dx\cdot\int_0^R e^{-y^2}dy \lim_{R\to+\infty}\iint_D e^{-(x^2+y^2)}dxdy, \tag{14}$$

Where $\mathrm{D}=[0,\mathrm{R}]\times[0,\mathrm{R}]$.

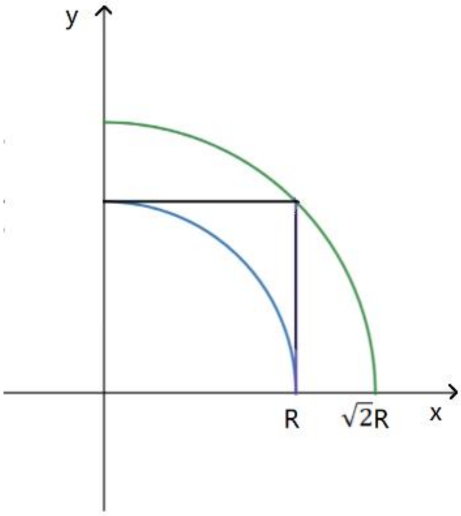

Figure 2. Illustration of pinch criterion

It is noted that the integrand function contains the square term, and the polar coordinate substitution is considered. However, the integral domain at this time is a square with side length R rather than a circular domain, which is difficult to be represented by polar coordinates, so it is considered to reduce the integral domain to a circular domain, and then use the pinch criterion [9]. As a result, the region is that

$$D_1: x^2+y^2\le R^2,\ D_2: x^2+y^2\le 2R^2 \tag{15}$$

In that case, let $x=\rho\cos\theta$ and $y=\rho\sin\theta$, with $0\le\rho\le R, 0\le\theta\le\frac{\pi}{2}$. Thus,

$$I_1=\lim_{R\to+\infty}\iint_{D_1} e^{-(x^2+y^2)}dxdy \tag{16}$$

Substitute the value of x, y in terms of ρ and θ, it is calculated that

$$\lim_{R\to+\infty}\iint_{D_1'}\rho e^{-\rho^2}d\rho d\theta=\lim_{R\to+\infty}\int_0^{\frac{\pi}{2}}d\theta\int_0^R\rho e^{-\rho^2}d\rho=\lim_{R\to+\infty}\frac{\pi}{4}(1-e^{-R^2})=\frac{\pi}{4}. \tag{17}$$

In the same way,

$$I_2=\lim_{R\to+\infty}\frac{\pi}{4}\left(1-e^{-R^2}\right)=\frac{\pi}{4}. \tag{18}$$

Considering the area of the integral domain, it can be get that

$$\iint_{D_1} e^{-(x^2+y^2)}dxdy < \iint_{D} e^{-(x^2+y^2)}dxdy < \iint_{D_2} e^{-(x^2+y^2)}dxdy \tag{19}$$

According to the pinch criterion, $I^2 = \frac{\pi}{4}$, $I=\frac{\sqrt{\pi}}{2}$. Hence, the proof for the identity

$$\int_0^{+\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2} \tag{20}$$

Has been accomplished.

3.2. Complex Analysis

Similarly, the main content of this session is calculating the general gaussian integral $\int_{-\infty}^{+\infty} e^{-x^2}dx$ in complex integral. The residue theorem will be applied to solve two different forms of Gaussian integral.

3.2.1. Normal form of Gaussian integral

For the integral

$$I = \int_{-\infty}^{+\infty} e^{-x^2}dx, \tag{21}$$

It is useful to introduce the function

$$f(z) = \frac{e^{-z^2}}{g(z)}. \tag{22}$$

Express the integration with the residue theorem,

$$\oint_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz + \int_{C_4} f(z)dz = 2\pi i \sum Res[f(z)]. \tag{23}$$

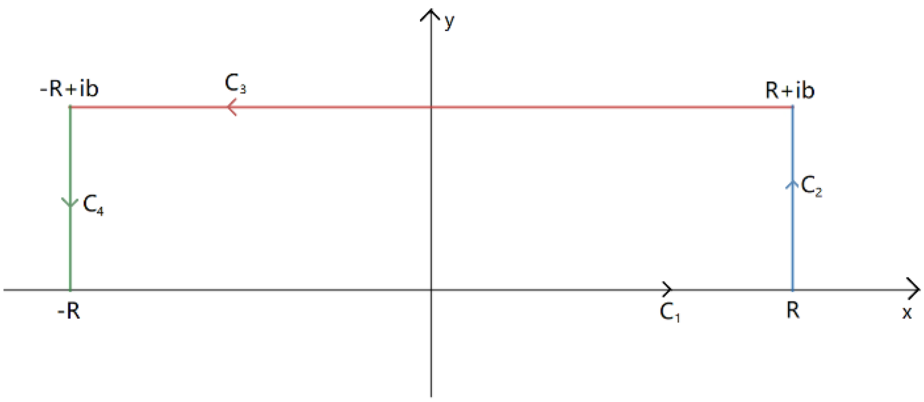

Figure 3. Illustration of the contour path

Consider C_1 and C_3 shown in Fig. 3, it is found that

$$\lim_{R\to+\infty} \int_{C_1} f(z)dz + \int_{C_3} f(z)dz = 2\pi i \sum Res[f(z)]. \tag{24}$$

Since $\int_{-R}^{R} f(z)dz + \int_{R+ib}^{-R+ib} f(z)dz = \int_{-R}^{R} f(z)dz - \int_{-R}^{R} f(z+ib)dz$, then

$$\lim_{R\to+\infty} \int_{-\infty}^{\infty} f(z) - f(z+ib)\, dz = 2\pi i \sum Res[f(z)]. \tag{25}$$

In the case that the equation $f(z) - f(z+ib) = e^{-z^2}$, it is easy to find that

$$f(z) - f(z+\tau) = e^{-z^2}. \tag{26}$$

On the basis of the equivalent relationship Eq. (21), one finds that

$$\frac{e^{-z^2}}{g(z)} - \frac{e^{-(z+\tau)^2}}{g(z+\tau)} = e^{-z^2}. \tag{27}$$

Assume that g is periodic in τ, then

$$\frac{e^{-z^2}}{g(z)} - \frac{e^{-z^2-2z\tau-\tau^2}}{g(z)} = \frac{e^{-z^2}}{g(z)} + \frac{e^{-z^2}e^{-z^2-2z\tau-\tau^2}}{g(z)} = e^{-z^2} \tag{28}$$

So $g(z) = 1 - e^{-2z\tau-\tau^2}$. Meanwhile, $g(z) = g(z+\tau)$, and this gives

$$1 - e^{-2z\tau-\tau^2} = 1 - e^{-2(z+\tau)\tau-\tau^2} = 1 - e^{-2z\tau-\tau^2-2\tau^2} = 1 - e^{-2z\tau-\tau^2}e^{-2\tau^2} \tag{29}$$

Now one can obtain that t$e^{-2\tau^2} = 1 = e^{-2\pi ik}, k\epsilon Z,\ -2\tau^2 = -2\pi ik,\ \tau = \sqrt{\pi ik}$.

Let $k = 1$, then $\tau = \sqrt{\pi i},\ \tau^2 = \pi i$. In other words, $\tau = \sqrt{\pi}e^{\frac{i\pi}{4}} = \sqrt{\pi}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right), \mathrm{Im}(\tau) = \sqrt{\frac{\pi}{2}}$. So $g(z) = 1 - e^{-2z\tau}e^{-\tau^2}1 - e^{-2z\tau}e^{-i\pi} = 1 + e^{-2z\tau}$. This gives that $f(z) = \frac{e^{-z^2}}{1+e^{-2z\tau}}$. For poles $1 + e^{-2z\tau} = 0$, and $e^{-2z\tau} = -1 = e^{i\pi(2k+1)}$ $(k = Z)$, $-2z\tau = i\pi(2k+1) = -2z\sqrt{\pi i}$, $z = \frac{\tau}{2}(2k+1)$, and $z = R + it, dz = idt$. Now the inequation is going to be obtained,

$$\left|\int_{C_2} f(z)dz\right| = \left|\int_{C_2} \frac{e^{-z^2}}{1+e^{-2z\tau}}dz\right| = \left|\int_0^{\sqrt{\frac{\pi}{2}}} \frac{e^{-(R+it)^2}}{1+e^{-2(R+it)\tau}}idt\right| \le \int_0^{\sqrt{\frac{\pi}{2}}} \frac{e^{-R^2+t^2}}{\left|1-\left|e^{-2\tau(R+it)}\right|\right|}dt \tag{30}$$

And

$$\int_0^{\sqrt{\frac{\pi}{2}}} \frac{e^{-R^2+t^2}}{\left|1-e^{-2\sqrt{\frac{\pi}{2}}R}e^{-2\sqrt{\frac{\pi}{2}}iit}\right|}dt = \int_0^{\sqrt{\frac{\pi}{2}}} \frac{e^{-R^2+t^2}}{\left|1-e^{-2\sqrt{\frac{\pi}{2}}R}e^{2\sqrt{\frac{\pi}{2}}t}\right|}dt. \tag{31}$$

Hence, $\lim\limits_{R\to+\infty}\left|\int_{C_2} f(z)dz \le 0\right| = \lim\limits_{R\to+\infty}\int_{C_2} f(z)dz = 0$.

Similarly, it can be calculated that

$$\lim_{R\to+\infty}\left|\int_{C_4} f(z)dz \le 0\right| = \lim_{R\to+\infty}\int_{C_4} f(z)dz = 0. \tag{32}$$

Thus, one can obtain that

$$\int_{C_1} f(z)dz + \int_{C_3} f(z)dz = \int_{-\infty}^{+\infty} e^{-x^2}dx = 2\pi i \operatorname*{Res}_{z=\tau/2}[f(z)] = \sqrt{\pi}. \tag{33}$$

3.2.2. Gaussian integral in form of $\int_0^{+\infty} e^{-ax^2}cosbx\, dxdx$

In this section, the author is going to use the residue theorem to deal with a less common but more general form. This particular form does appear in some situations. Obviously, this form will be more complex and difficult to deal with, but it will also give a deeper understanding of Gaussian integrals and probability density functions. At the same time, the residue theorem greatly helps people to deal with this complicated situation [10].

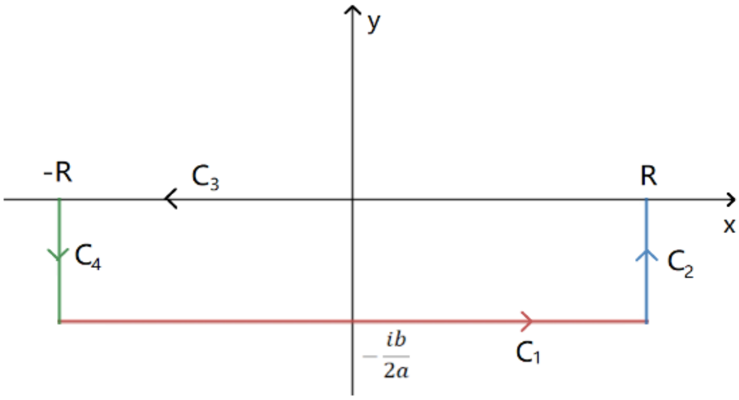

Figure 4. Illustration of the contour path

For the Gaussian integral in the form of

$$J = \int_0^{+\infty} e^{-ax^2} cosbx\, dxdx, a > 0, b\epsilon R. \tag{34}$$

To begin with, it is found that $e^{-ax^2} cosbx = Re\, e^{-ax^2+bx} = e^{\frac{b^2}{4a}} Re\, e^{-a\left(x^2-\frac{ib}{2a}x\right)}$. Let $f(z) = e^{-az^2}$ is analysis on C (for illustration, see Fig. 4), then

$$I = \int_{-\infty-\frac{ib}{2a}x}^{+\infty-\frac{ib}{2a}x} e^{-az^2} dz = \int_{-R-\frac{ib}{2a}x}^{R-\frac{ib}{2a}x} e^{-az^2} dz = \int_{C_1} e^{-az^2} dz. \tag{35}$$

By virtue of the Cauchy lemma, it is arrived that

$$\int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz + \int_{C_4} f(z)dz = 0, \tag{36}$$

Where C_2: $z = R + iy, -\frac{b}{2a} \le y \le 0,\ z^2 = R^2 - y^2 + 2Ryi$. Therefore,

$$\int_{C_2} e^{-az^2} dz = \int_{-\frac{b}{2a}}^{0} e^{-a(R^2-y^2)-2aRyi} dy \tag{37}$$

And

$$\left|\int_{C_2} e^{-az^2} dz\right| = \left|\int_{-\frac{b}{2a}}^{0} e^{-a(R^2-y^2)-2aRyi} dy\right| \le \int_{-\frac{b}{2a}}^{0} e^{-aR^2+ay^2} dy = e^{-aR^2} \int_{-\frac{b}{2a}}^{0} e^{ay^2} dy. \tag{38}$$

Hence, $\lim_{R\to+\infty} e^{-aR^2} \int_{-\frac{b}{2a}}^{0} e^{ay^2} dy = 0$ and $\lim_{R\to+\infty} \int_{C_4} e^{-az^2} dz = 0$. Similarly,

$$\int_{C_3} e^{-az^2} dz = -\int_{-R}^{R} e^{-ax^2} dx = -\sqrt{\frac{x}{a}}. \tag{39}$$

By virtue of the relevant equations shown above, it is found that

$$\int_0^{+\infty} e^{-ax^2} cosbx\, dxdx = \frac{1}{2}(Re\, I) e^{-\frac{b^2}{4a}} = \frac{1}{2}\sqrt{\frac{\pi}{a}} e^{-\frac{b^2}{4a}}. \tag{40}$$

4. Conclusion

In summary, this paper takes Gaussian integral as an example to do a series of studies on the real analysis and complex analysis of probability density function. It is a popular trend to extend the probability density coefficient to the complex plane in recent years. In fact, probability density functions such as the Gaussian integral were once thought to be unsolvable by the residue theorem, because the Gaussian integral appears to be analytic in all planes. However, this research method has become one of the hottest directions in the academic world, and many papers are published on this aspect every year. There is no doubt that the methods of real analysis and complex analysis have complementary properties. As stated in the introduction, real and complex analysis of the probability density function can first give people a more complete understanding of the function, and a deeper understanding of its properties and behavior. Complex analysis provides a tool to study the probability density function on the complex plane, and can reveal the periodic, singular, and extreme points of the function. Real analysis, on the other hand, focuses on the properties of functions over real numbers, such as continuity and derivability. At the same time, by integrating complex analysis and real analysis, one can solve a wider range of problems. Complex analysis can be used to solve the Fourier transform and Laplace transform of probability density function, which is very useful for signal processing and filter design in probability theory and statistics. The real analysis can be used to solve the integral and differential equations of probability density function, which is very important for calculating probability and deriving the properties of probability density function. In general, comprehensive complex analysis and real analysis can provide more comprehensive and in-depth methods and tools to study and understand probability density function, and expand the application field of probability theory and statistics. Of course, in the future, the biggest beneficiaries of this research approach will certainly be probability and statistics. However, this paper only studies a typical integral, Gaussian integral, which is not representative enough to some extent. In some sense, the scope of the research still has room to expand, and the actual application and feasibility need to be further studied.

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