

Definition of Residue Theorem and its Basic Applications

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Abstract. This paper explains the fundamentality of residue theorem in complex analysis integration termed contour integral. It also illustrates the importance of this theorem, followed by applying it in both mathematics and other fields where several typical examples coming from different fields are chosen. The paper introduces some basic definitions and theorems to support and finally prove residue theorem. Then two simple applications of residue theorem are presented. One is a contour integral of a fractional function with one singularity inside the contour. It can be solved by using partial fraction technique, then directly finding the residue at the singularity, and finally applying residue theorem to calculate for the result. The other one is a contour integral of a reciprocal of sine function with one singularity inside. This problem can be solved by finding the Laurent series of the integrand, thus finding the residue needed. However, the residue is the coefficient of negative-one-degree term of the singularity, and the value of the integral can be achieved by substituting the residue into the formula of the residue theorem.

Keywords: Residue theorem, contour integral, Laurent series.

1. Introduction

Calculus of real functions is well-known and widely studied in university, but that of complex functions is also a deep, complicated topic that requires careful investigation. Being simple yet fundamental, Cauchy's residue theorem is a powerful tool and the basis of a huge part of complex analysis. As explained in Ref. [1], residue theorem combines Cauchy's integral theorem, Cauchy's integral formula, and derivatives of high order formula. Researchers have continually made extensions or further discussions of residue theorem, which shows the importance of this theorem and researchers' constant interest on it. Since residue theorem is a basic theorem in complex calculus, the ability to proficiently applying it to problems on complex calculus is vital for keeping exploring on this field, and this theorem is worth serious study and investigation.

The usage of residue theorem must not be ignored. Residue theorem can be used for real functions; it can help find some integral of real functions and the sum of series [2], and it can be applied to bilateral hypergeometric series, which is composed of real functions [3]. It can also be used to derive matrix functions, which is a novel and simple way [4]. Furthermore, residue theorem can be applied to fields other than mathematics. The result for second order displacement of one-dimensional Su-Schrieffer-Heeger model can be directly calculated using residue theorem [5]. Residue theorem plays an important role on wind energy problems, setting up and solving an equation of mechanical speed and thus control the wind speed generated effectively [6]. Residue theorem can be used to analyze electrical networks with exponential source, and this method using residue theorem is a "simple and effective way" [7].

In what follows, this paper introduces several basic theorems and definitions, including contour integral, Cauchy-Goursat theorem, singularity, Laurent series, and residue, to support and prove residue theorem. Subsequently, two applications of residue theorem to simple contour integral problems are presented, respectively on a fractional function and on a sine function, which represents trigonometric function.

2. Definitions and Theorems

2.1. Contour Integral

Contour integral is an arithmetic in calculus. A simple curve C drawn on the complex plane is a contour; then a contour integral of function $f: \mathbb{C} \rightarrow \mathbb{C}$ can be set on this contour, written as $\int_C f(z) dz$. If $z = z(t)$, where $t \in \mathbb{R}$ and $a \leq t \leq b$ making sure the range of z is C defined above, Chain Rule can be used to acquire $dz = z'(t)dt$ and

$$\int_C f(z) dz = \int_a^b (z(t))z'(t) dt. \quad (1)$$

Here, this integral starts from a and ends at b . If they are exchanged, the result of the integral will be negative. Therefore, contour C has a direction, and it is defined to be positively oriented if it is counterclockwise.

Contour integral works similarly to linear integral but it is “curved”. A linear integral is dependent on starting and ending values, and sometimes a contour integral is independent on the shape of the contour but only the starting and ending points, so the notation of contour can be replaced as $\int_{z_1}^{z_2} f(z) dz$, only when the integral is not dependent on the shape of contour. The value of the integral may be dependent on the contour even between fixed starting and ending points because of singularities inside the contour.

2.2. Cauchy-Goursat Theorem

If the starting point and the ending point of a real integral are the same, the result will be 0 because the area covers across a width of 0. In contour integration, this situation means a contour of a closed curve, so the integral is still along a path with certain length and thus the result will be subtle. The Cauchy-Goursat theorem states that if function f is analytic at all points in and on simple closed contour C , or it is analytic throughout a simply connected domain which a closed contour C is in, $\int_C f(z) dz = 0$. The proof of this theorem is explained in Ref. [8].

This theorem can also be adapted to multiply connected domain. For positively oriented simply closed contours C and C_k ($k = 1, 2, \dots, n$) that are inside C and disjoint with any other, if function f is analytic on the closed multiply connected domain inside C but outside C_k and including C and C_k , then

$$\int_C f(z) dz - \sum_{k=0}^n \int_{C_k} f(z) dz = 0. \quad (2)$$

2.3. Singularity

If a function is not analytical, that is, it is not defined or its derivative does not exist at a place, there is a singularity. If a function is analytical around a point but is not at the point, this point is an isolated singularity. There are three kinds of isolated singularity: removable singularity, pole, and essential singularity.

2.4. Residue

Residues of a function about a point can be found using Laurent series. Taylor series is a way to express a function in a polynomial at a point, written as

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad (3)$$

Laurent series, in contrast, contains also fractions

$$f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n + \sum_{n=1}^{\infty} b_n(x - x_0)^{-n}. \tag{4}$$

Laurent series can only be expressed at singularities, or it will reduce to Taylor series. The residue of a function at a point is the b_1 in expression above, which is the coefficient of $\frac{1}{z-z_0}$ term in Laurent series. As expressed in Laurent’s theorem,

$$b_1 = \frac{1}{2\pi i} \int_C f(z) dz \tag{5}$$

For C a positively rotated simple closed contour in an annular domain $R_1 < |z - z_0| < R_2$ which f is analytic throughout. The residue of function f at z is expressed as $\text{Res}f(z)$ or $\text{Res}(f, z_0)$. The residue at $\pm\infty$ can be calculated from the residue at 0 [9].

2.5. Residue Theorem

If function f is analytic in and on positively oriented simple closed contour C except for finitely many singularities inside (z_1, z_2, \dots, z_n) , then

$$\int_C f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}(f, z_k). \tag{6}$$

This is because if separate positively oriented contours C_1, C_2, \dots, C_n are made containing their according singularities. Note that C_k contains z_k only, for $1 \leq k \leq n$, see Fig. 1. They do not touch any other contour including C . Thus, Eq. (2) can be used to describe this situation

$$\int_C f(z) dz - \sum_{k=0}^n \int_{C_k} f(z) dz = 0. \tag{7}$$

Eq. (5) can be transformed to

$$2\pi i \text{Res}(f, z_k) = \int_{C_k} f(z) dz \tag{8}$$

Combined with Eq. (7) and Eq. (8), the residue theorem in Eq. (6) is proved. In such approach to contour integral, the problem of evaluating one or more contour integrals is replaced by the algebraic problem of computing residues at the enclosed singular points [10].

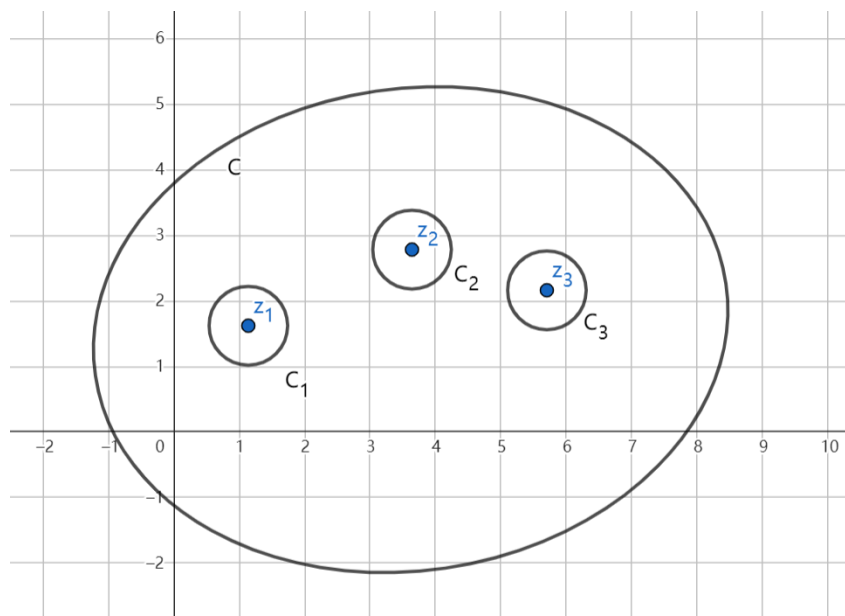


Figure 1. Graph of closed contours and singularities inside

3. Applications

3.1. Fractional Functions

The use of residue theorem in solving integration of fractional functions can reduce the effort needed and improve efficiency. To begin with, this paper tends to solve following integral

$$I = \int_C \frac{1}{z(z-2i)} dz \tag{9}$$

Where simple closed contour C is a unit circle centered at the origin.

This problem can be solved by substituting $z = e^{i\theta}$ ($0 \leq \theta \leq 2\pi$) and directly using the definition of contour integral, Eq. (1). However, anti-derivative is needed for this method, which is complex and sometimes even unsolvable by hand. Residue theorem can be used instead, since the integrand is analytical on and interior of the contour except at point $z = 0$. Therefore, according to Eq. (6), it is found that

$$\int_C \frac{1}{z(z-2i)} dz = 2\pi i \operatorname{Res}\left(\frac{1}{z(z-2i)}, 0\right). \tag{10}$$

There are multiple ways to calculate the residue of the integrand at 0, and a plain way is partial fraction. Set complex numbers A and B , there must be unique values for them satisfying that

$$\frac{A}{z} + \frac{B}{z-2i} = \frac{1}{z(z-2i)}. \tag{11}$$

Here, $\frac{B}{z-2i}$ can be expressed with Taylor series at 0. Since no extra $\frac{1}{z-0}$ term appears, so A is the only coefficient of $\frac{1}{z-0}$ term. This quantity can be treated as the residue of the integrand at 0, so

$$\int_C \frac{1}{z(z-2i)} dz = 2\pi i A. \tag{12}$$

After solving Eq. (11), $A = \frac{i}{2}$ will be obtained. This result can be substituted into Eq. (12) and finally solve the original problem

$$\int_C \frac{1}{z(z-2i)} dz = 2\pi i \frac{i}{2} = -\pi. \tag{13}$$

3.2. Trigonometric Functions

Trigonometric functions can be hard to directly calculate, but they are easy to analyze if using Taylor series of them as tool. With the help of Taylor series, residue theorem also works well on trigonometric functions. The problem at hand is

$$I = \int_C \frac{1}{\sin z} dz, \tag{14}$$

Where the closed contour C represents a unit circle centered at the origin.

This integrand is only not defined at point $z = 0$, so the residue of the integrand at 0 is needed. A Laurent series becomes a Taylor series after multiplying the variable to the power of opposite number of the least exponent of the Laurent series. Therefore, if a Taylor series can be created in that way, certain term of it will contain the residue of the original function. Finding the possible Taylor expansion of the integrand will thus be the first step. Set a Taylor series that satisfies $a_0 + a_1x +$

$a_2x^2 + \dots = \frac{1}{\sin x}$, where a_k ($k \geq 0, k \in \mathbb{Z}$) are constants. Equivalently, $(a_0 + a_1x + a_2x^2 + \dots) \sin x = 1$. Since the Taylor series of sine function is known, then

$$(a_0 + a_1x + a_2x^2 + \dots) \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right) = 1. \quad (15)$$

This equation is not possible since the least possible exponent on the left side is 1 (the a_0x term), but the right side is a constant. Therefore, if the original integrand is multiplied by x , this equation will make sense. Namely, $a_0 + a_1x + a_2x^2 + \dots = \frac{x}{\sin x}$ and $(a_0 + a_1x + a_2x^2 + \dots) \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right) = x$. This forms a possible Taylor expansion for $\frac{x}{\sin x}$, and the coefficient of term x on the left side should be the same as that on the right side, i.e., $a_0 = 1$. The original integrand is $\frac{x}{\sin x}$ divided by x , so the Laurent expansion of it must be $\frac{a_0}{x} + a_1 + a_2x + a_3x^2 \dots$. Obviously, $a_0 = 1$ is now the coefficient of $\frac{1}{x-0}$, so $\text{Res} \left(\frac{1}{\sin z}, 0 \right) = 1$. Consequently,

$$\int_C \frac{1}{\sin z} dz = 2\pi i \text{Res} \left(\frac{1}{\sin z}, 0 \right) = 2\pi i. \quad (16)$$

4. Conclusion

The application of Cauchy's residue theorem, of which two examples on fractional functions and trigonometric functions are shown above in this paper, is important for explorers on this field of complex integration to understand. The proficiency in techniques used in those examples, such as partial fractions, Taylor series, and the Laurent series derived from Taylor series, and other techniques means a way of thinking for solving math problems, or specifically complex integrals. With this way of thinking, one can practice, consider, and solve math problems and develop in the field of mathematics. In addition, those techniques are used for obtaining residues and finally applying residue theorem in this paper. Residue theorem elegantly uses several singularities of a function inside a contour to calculate for normally complicated integral. This simplicity makes residue theorem a practical approach to many hard, previously unsolvable problems. Therefore, residue theorem is a basic topic in complex analysis, it brings out various topics and discussions in this field, and it is worth study and discussion on it. Residue theorem has a much wider range of application than explained in this paper. Many other complex integrals can be solved using it, like that of exponential functions, logarithm functions, functions of combination or nest of different types, non-elementary functions, etc. The application of residue theorem also covers various fields other than mathematics. As explained in Introduction, many physics problems, environmental science and energy-related problems, and other complicated models can be simplified using residue theorem. Familiarizing residue theorem is beneficial or even vital for further research and study, and the purpose of this paper is to help in this regard.

References

- [1] Zhang Kunshi. Residue Theorem and Integral of Complex Variable Function. Journal of Higher Correspondence Education (Natural Sciences), 2003, 16 (1): 13 - 17.
- [2] Xu Ning. Method of the Complex Analysis to Some Series. PLA Nanjing Institute of Politics, 2014, 37 (4): 20 - 27.
- [3] Chu Wenchang, Wang Xiaoxia, Zheng Deyin. Application of the Residue Theorem to Bilateral Hypergeometric Series. Le Matematiche, 2007, LXII (II): 127 - 146.

- [4] Liu Zhixiu, Huang Xiaojie, Jin Benqing, Xie Jiehua. The Application of Residue Theorem in Calculation of Matrix Function Value. *Journal of Jiangxi University of Science and Technology*, 2013, 34 (5): 96 - 99.
- [5] Meng Ya, Guan Xin. Application of the Residue Theorem in Topological Phase Transitions. *College Physics*, 2023, 42 (1): 7 - 13.
- [6] Alsumiri, M., Li, L., Jiang, L. et al. Residue Theorem Based Soft Sliding Mode Control for Wind Power Generation Systems. *Prot Control Mod Power Syst*, 2018, 3 (24).
- [7] Rohit Gupta, Loveneesh Talwar, Dinesh Verma. Exponential Excitation Response of Electric Network Circuits via Residue Theorem Approach. *International Journal of Scientific Research in Multidisciplinary Studies*, 2020, 6 (3): 47 - 50.
- [8] Brown, J. W., & Churchill, R. V. (2014). *Complex Variables and Applications*. McGraw-Hill Education.
- [9] Hanqi Zhang, Yifan Chen, Jicheng Tao. The Theory of Residue for Heterogeneous Complex Variables Functions. *Advances in Applied Mathematics*, 2021, 10 (2): 587 - 597.
- [10] Arfken G. B., Weber, H.-J. *Mathematical Methods for Physicists*. Elsevier Acad. Press, 2011.