Existence of Solutions for a Class of Nonlinear Convolution Integral Equations

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Abstract. The existence, uniqueness, boundedness and monotonicity of solutions for a class of nonlinear convolutional integral equations are discussed. First, the problem of solving the equation is transformed into a fixed-point problem of an operator. Then, Arzela – Ascoli theorem and Schauder fixed point theorem are used to prove the existence of the solution. Then, Gronwall inequality and its related lemma are used to prove the uniqueness of the solution. Secondly, the sufficient and necessary conditions for boundedness of nonnegative solutions are given by using the supremum principle and Weierstrass aggregation point theorem. Finally, under given conditions, the monotonicity of the nonnegative solution is discussed, which extends the existing results.

Keywords: Convolutional integral equation, Existence and uniqueness of solutions, Boundedness, Monotonicity.

1. Introduction

Consider the following class of nonlinear convolutional integral equation

$$\Phi(u(t)) = L(t) + \int_0^t P(t-s)u^m(s)\mathrm{d}s, \quad t \ge 0 \tag{1}$$

Where $L(t)$ and $P(t)$ are continuous positive functions on the interval $[0,+\infty)$ and $u(t)$ is unknown positive functions, Set Φ to meet the following conditions:

i) Φ is a continuous function of $[0,+\infty) \to [0,+\infty)$, and is a locally expansive on $[0,+\infty)$, that is, for any compact set $K \subset (0,+\infty)$, there exists a constant $c_k > 0$ such that

$$\begin{gathered} \left|\Phi(u_1) - \Phi(u_2)\right| \ge c_k \left|u_1 - u_2\right|, \\ u_1, u_2 \in K \,\&\, u_1 \ne u_2 \end{gathered} \tag{2}$$

ii) $\lim\limits_{u\to\infty} \dfrac{\Phi(u)}{u^m} = \infty$

iii) $\lim\limits_{u\to 0} \dfrac{\Phi(u)}{u^m} = 0$

In case $\Phi(u) = u, m = 1$, equation (1) has been fully discussed in population theory, industrial substitution theory and polymer theory [1, 2]. The explicit solution of the equation in this case can be obtained by means of Laplace transform, and the asymptotic behavior of the solution is studied [3]. In case $\Phi(u) = u^2, m = 1$, the equation is the seepage model of water in impermeable porous media, and its solution $u(t)$ is the height of water at time t [4, 5]. Therefore, it is of great practical significance to study the nonnegative solution of the equation. Many scholars have proved that the equation has a unique nonnegative solution [2] [5] [6] [7], and the boundedness of the nonnegative solution has also been discussed [4] [8] [9] [10]. Although there is no explicit solution for the equation under the nonlinear condition, O. Lipovan [3] [11] gave a finite condition to ensure the existence of $\lim\limits_{t\to\infty} u(t)$ for the case of $\Phi(u) = u^2, m = 1$ and the case of $\Phi(u) = u^p, p > 1, m = 1$ .In the case that function Φ satisfies the conditions i) - iii), $m=1$, O. Lipovan [12] proved that if $L\in(R_+,(0,\infty))$, $P,\Phi\in(R_+,R_+)$ and $P\not\equiv 0$ equation (1) have a unique non negative solution, and gave the necessary and sufficient condition that the non-negative solution is bounded if and only if $\int_0^\infty P(t)\mathrm{d}t<\infty$ and L are bounded, O. Lipovan, S. Jack and Hargraves [13] proved that if function Φ is continuously differentiable and satisfies the conditions i)-iii), L,P is continuously differentiable and P is a nonincreasing function, when $m=1$, if $L'(t)+U_0P(t)>0$ for all $t\in[0,\infty)$, $u(t)$ is strictly increasing on $[0,+\infty)$.

The specific work steps are as follows: In the first part, on the basis of the existing of [12], it is proved that when $m>0$, equation (1) has a unique nonnegative solution. In the second part, we discuss the boundedness of the nonnegative solution of equation (1), and give the necessary and sufficient conditions for the boundedness of the nonnegative solution to be $\int_0^\infty P(t)\mathrm{d}t<\infty$ and L is bounded. In the third part, we improve the conditions [13] and prove that when $m>0$, , if $L'(t)+U_0^mP(t)>0$ for all $t\in[0,\infty)$, then $u(t)$ is strictly increasing on $[0,+\infty)$.

2. Discussion on Existence and Uniqueness

Lemma 1.1 [14] In order that $F\subset C(M)$ is a sequentially compact set, F must and only must be a family of uniformly bounded and equicontinuity functions.

Lemma 1.2 [14] Let C be a closed convex subset of B^* spaces X, $T:C\to C$ continuous and $T(C)$ compact, then T must have a fixed point on C.

Lemma 1.3 [15] Let $u,f\in C([t_0,T),\mathrm{R}_+)$, $\omega\in C(\mathrm{R}_+,\mathrm{R}_+)$ and $\alpha\in C^1([t_0,T],[t_0,T])$ are non-increasing functions, $\omega(u)>0$ on $(0,\infty)$ and $\alpha(t)\le t$ on $[t_0,T)$. if

$$u(t)\le k+\int_{\alpha(t_0)}^{\alpha(t)} f(s)\omega(u(s))\mathrm{d}s, t_0\le t\le T, \tag{3}$$

Where k is a nonnegative constant, for any $t_1\in(t_0,T)$, if $t_0\le t\le t_1$ then

$$u(t)\le G^{-1}\left(G(k)+\int_{\alpha(t_0)}^{\alpha(t)} f(s)\mathrm{d}s\right), \tag{4}$$

There exists a function $G(r)=\int_1^r \frac{\mathrm{d}s}{\omega(s)}, r>0$ such that

$$G(k)+\int_{\alpha(t_0)}^{\alpha(t)} f(s)\mathrm{d}s\in Dom(G^{-1}) \tag{5}$$

For all $t\in[t_0,t_1)$.

Lemma 1.4 [15] Consider the following integral equation

$$u(t)=a\int_0^t \omega(u(s))\mathrm{d}s, t\ge 0 \tag{6}$$

Where $\omega\in C(\mathrm{R}_+,\mathrm{R}_+), \omega(0)=0$, a is a constant, for any function ω, suppose that

$$|\omega(x)-\omega(y)|\le z|x-y|, x,y\in\mathrm{R}_+ \tag{7}$$

Where $z\in C(\mathrm{R}_+,\mathrm{R}_+)$ is a non-subtractive function, and for all $x>0$, there $z(x)>0$. if

$$\int_0^1 \frac{ds}{z(s)} = \int_1^\infty \frac{ds}{z(s)} = \infty \tag{8}$$

Then equation (6) has a unique nonnegative solution.

Theorem 1.1 Let $L \in C(R_+,(0,\infty)), P,\Phi \in C(R_+,R_+)$, assume that function Φ satisfies the condition i)-iii), then when $m>0$, equation (1) has a unique nonnegative solution.

Proof. Set $T>0$ and set

$$P_T = \sup_{t\in[0,T]} P(t), L_T = \sup_{t\in[0,T]} L(t) \tag{9}$$

Schauder fixed point theorem is applied to this operator.

$$Su(t) = \Phi^{-1}\left(L(t) + \int_0^t P(t-s)u^m(s)ds\right), \quad t \ge 0, m > 0 \tag{10}$$

Defining Bounded Closed Convex Sets

$$B = \left\{u:[0,T] \to R_+ \text{continuity}, \|u\| = \sup_{t\in[0,T]} |u(t)| \le c\right\} \tag{11}$$

We can see that by the condition ii) of Φ, there exist some $c>1$ satisfying

$$L_T + P_T \cdot T \cdot c^m \le \Phi(c). \tag{12}$$

Here we can prove that $S(B) \subseteq B.$ let $u \in B,$ by using the monotonicity of Φ^{-1} and formula (12), it can be derived that

$$Su(t) \le \Phi^{-1}\left(L_T + P_T \cdot T \cdot c^m\right) \le c, \tag{13}$$

Which shows that $S(B) \subseteq B.$

Now we prove that S is compact. Therefore, select a sequence $\{u_n\} \subset B.$ then $\|Su_n\| \le c,$, that is, family $\{Su_n\}$ is uniformly bounded.

Let us prove that family $\{Su_n\}$ is equicontinuous. Let $s,t \in [0,T], s \ge t.$ by the condition i) of Φ, there is a constant $c_T > 0$ such that

$$\left|\Phi(u_n(s)) - \Phi(u_n(t))\right| \ge c_T \, | u_n(s) - u_n(t) |, \tag{14}$$

Thus

$$| u_n(s) - u_n(t) | \le \frac{1}{c_T}\left|\Phi(u_n(s)) - \Phi(u_n(t))\right|, \tag{15}$$

Since $u(t) = \Phi^{-1}\left(L(t) + \int_0^t P(t-s)u^m(s)ds\right),$ set $h = \frac{1}{c_T},$ it can be obtained from the above formula

$$\begin{aligned}&\left|\Phi^{-1}\left(L(s) + \int_0^s P(s-r)u_n^m(r)dr\right) - \Phi^{-1}\left(L(t) + \int_0^t P(t-r)u_n^m(r)dr\right)\right| \\ &\le h\left|\Phi(u_n(s)) - \Phi(u_n(t))\right|\end{aligned} \tag{16}$$

And because of $\inf_{t\in[0,T]} L(t) > 0$, so

$$\begin{aligned}
&\left|Su_n(s)-Su_n(t)\right| \\
&=\left|\Phi^{-1}\left(L(s)+\int_0^s P(s-r)u_n^m(r)\mathrm{d}r\right)-\Phi^{-1}\left(L(t)+\int_0^t P(t-r)u_n^m(r)\mathrm{d}r\right)\right| \\
&\le h\left(|L(s)-L(t)|+\left|\int_0^s P(s-r)u_n^m(r)\mathrm{d}r-\int_0^t P(t-r)u_n^m(r)\mathrm{d}r\right|\right) \\
&\le h\left(|L(s)-L(t)|+\int_0^t\left|P(s-r)-P(t-r)\right|u_n^m(r)\mathrm{d}r+\int_t^s P(s-r)u_n^m(r)\mathrm{d}r\right) \\
&\le h\left(|L(s)-L(t)|+c^mT\left[\sup_{r\in[0,T]}\left|P(s-r)-P(t-r)\right|\right]+P_Tc^m\left|s-t\right|\right)
\end{aligned} \tag{17}$$

It is known that P is a continuous positive function on $[0,+\infty)$, so P is continuous on $[0,T]$, it is uniformly continuous on $[0,T]$. And because the right end of the above inequality is independent of n, when $|s-t|\to 0$, the right end of the above equation tends to zero, which proves that family $\{Su_n\}$ is equicontinuous. According to lemma 1.1, S is compact. Then from lemma 1.2, we can see that S must have a fixed point in B, so equation (1) has a nonnegative solution on [0, T].

To prove that the nonnegative solution is unique. Let u_1, u_2 be the nonnegative solution of equation (1) on $[0,T]$, since $\inf_{t\in[0,T]} L(t) > 0$, we deduce

$$\begin{aligned}
&|u_1(t)-u_2(t)| \\
&=\left|\Phi^{-1}\left(L(t)+\int_0^t P(t-r)u_1^m(r)\mathrm{d}r\right)-\Phi^{-1}\left(L(t)+\int_0^t P(t-r)u_2^m(r)\mathrm{d}r\right)\right| \\
&\le hP_T\int_0^t\left|u_1^m(r)-u_2^m(r)\right|\mathrm{d}r.
\end{aligned} \tag{18}$$

Now we prove that there is a constant $\lambda > 0$ such that $|u_1^m(r)-u_2^m(r)| \le \lambda |u_1(r)-u_2(r)|$.

Let $M = \max\left\{\max_{[0,T]} u_1(t), \max_{[0,T]} u_2(t)\right\}, X\in[0,M], Y\in[0,M],$

Fix $m \ge 1$, by the differential mean value theorem, we deduce

$$|X^m - Y^m| \le mM^{m-1}|X-Y| \tag{19}$$

Let $\lambda = mM^{m-1}$ in relation (19), this implies that there is a constant $\lambda > 0$ such that $|u_1^m(r)-u_2^m(r)| \le \lambda |u_1(r)-u_2(r)|$.

Now consider the case of $0<m<1$. Let

$M_1 = \min\left\{\min_{[0,T]} u_1(t), \min_{[0,T]} u_2(t)\right\}, X_1\in[0,M_1], Y_1\in[0,M_1],$

Since $0<m<1$, so $\frac{1}{m} > 1$. by the differential mean value theorem, we obtain that

$$\left|X_1-Y_1\right| = \left|\left(X_1^m\right)^{\frac{1}{m}}-\left(Y_1^m\right)^{\frac{1}{m}}\right| \ge \frac{1}{m}\left(M_1^m\right)^{\frac{1}{m}-1}\left|X_1^m-Y_1^m\right|. \tag{20}$$

Which snows that $|X_1^m - Y_1^m| \le mM_1^{m-1}|X_1-Y_1|$.

Let $\lambda = mM_1^{m-1}$ in the above formula, we can see that there exicts a constant $\lambda > 0$ such that

$|u_1^m(r)-u_2^m(r)|\le\lambda|u_1(r)-u_2(r)|.$

Now we proof that there exicts a constant $\lambda>0,$ when $m>0$, there is

$|u_1(t)-u_2(t)|\le hP_T\int_0^t\left|u_1^m(r)-u_2^m(r)\right|\mathrm{d}r\le\lambda hP_T\int_0^t|u_1(r)-u_2(r)|\mathrm{d}r.$

Let function $\omega(x)=x$ have $\forall x>0$ for $z(x)=\frac{1}{x}+1$ so that

$$|\omega(x)-\omega(y)|\le z(|x-y|),x,y\in\mathrm{R}_+, \tag{21}$$

Then $|\omega(u_1(r))-\omega(u_2(r))|\le z(|u_1(r)-u_2(r)|).$

Fix $T>0,$ set $G(T)=\int_1^T\frac{\mathrm{d}r}{z(r)},$ we deduce $G(0)=\infty$

And $G(\infty)=\infty.$ let $a=\lambda hP_T,$ we have

$$|u_1(t)-u_2(t)|\le a\int_0^t z\left(|u_1(r)-u_2(r)|\right)\mathrm{d}r, \tag{22}$$

In view of Lemma 1.4 we know that $|u_1(t)-u_2(t)|=0,$ thus $u_1(t)=u_2(t),$ this shows that equation (1) has a unique nonnegative solution.

3. Discussion of boundedness

Theorem 2.1 Let $L\in C\left(R_+,(0,\infty)\right),P,\Phi\in C\left(R_+,R_+\right)$ and $P\not\equiv 0,$ function Φ satisfies conditions i) - iii), then when $m>0$, the solution $u(t)$ of equation (1) is bounded if and only if $\int_0^\infty P(t)\mathrm{d}t<\infty$ and L is bounded.

Proof. Let $I=\int_0^\infty P(t)\mathrm{d}t<\infty$, L be bounded.

According to the supremum principle, if L is bounded, then L must have a supremum. Let $\sup\limits_{t\ge0}L(t)=K<\infty.$

Let $M_T=\sup\limits_{0\le t\le T}u(t),$ for any $T>0,$ from equation (1) we obtain

$$\begin{aligned}\Phi(M_T)&\le K+\sup_{0\le t\le T}\int_0^t P\left(t-s\right)u^m(s)\mathrm{d}s\\&\le K+M_T{}^mI.\end{aligned} \tag{23}$$

It can be seen from condition ii) that there is a constant $\delta=\delta(K,I)$, so that when $x>\delta$, there is

$$\Phi(x)>K+x^mI \tag{24}$$

Therefore $M_T<\delta$, since $T>0$ is arbitrary, $u(t)$ has an upper bound on R_+.

It can be seen from the proposition that $u(t)$ is an unknown positive function, so $u(t)>0$ means that $u(t)$ has a lower bound on R_+, this proves $u(t)$ is bounded on R_+.

Conversely, assume that solution $u(t)$ of equation (1) is bounded on R_+.

Because $L(t)$ and $P(t)$ are continuous positive functions on $[0,+\infty)$, then $L(t)$ and $P(t)$ have lower bounds on R_+ .It can be seen from equation (1) that when $t \geq 0$, there is $\Phi(u(t)) \geq L(t)$. Obviously, L is bounded on R_+, so L is bounded on R_+.

According to Weierstrass aggregation theorem, $u(t)$ and $L(t)$ have at least one aggregation point on R_+, and there are maximum aggregation points and minimum aggregation points. Current definition

$$a = \liminf_{t\to\infty} u(t), l = \liminf_{t\to\infty} L(t), \tag{25}$$

Take $T > 0$ of any $\varepsilon > 0$, so that when $t \geq T$, there are $u(t) \geq a - \varepsilon.$ Note that

$$\begin{aligned}\int_T^t P(t-s)u^m(s)\mathrm{d}s &\geq (a-\varepsilon)^m \int_T^t P(t-s)\mathrm{d}s \\ &= (a-\varepsilon)^m \int_0^{t-T} P(s)\mathrm{d}s, t \geq T\end{aligned} \tag{26}$$

Select a sub column $u(t_n)$, when $t_n \to \infty, u(t_n) \to a$. In equation (1), order $t = t_n$ to obtain

$$\Phi(u(t_n)) = L(t_n) + \int_0^{t_n} P(t_n - s)u^m(s)\mathrm{d}s, \quad t_n \geq 0. \tag{27}$$

Then when $n \to \infty$, it is obtained from (26) and the inequality preserving property of the lower limit

$$\Phi(a) \geq l + (a-\varepsilon)^m I. \tag{28}$$

First, set $a > 0$. Obviously, for all $\varepsilon > 0$, the above equation is true. Now let $\varepsilon = \frac{a}{2}$, $I < \infty$ can be deduced from (26).

Now consider the case of $a = 0$. Because when $t \geq 0$, there is $\Phi(u(t)) \geq L(t) > 0$. Therefore, if $a = 0$, we can find a sub column $t_n \to \infty$, such that

$$\inf_{t\in[0,t_n]} u(t) = u(t_n) \to 0 \tag{29}$$

When $t = t_n$, it can be deduced from (1) that,

$$\begin{aligned}\Phi(u(t_n)) &\geq \int_0^{t_n} P(t_n - s)u^m(s)\mathrm{d}s \\ &\geq u^m(t_n)\int_0^{t_n} P(t_n - s)\mathrm{d}s = u^m(t_n)\int_0^{t_n} P(s)\mathrm{d}s.\end{aligned} \tag{30}$$

Thus,

$$\frac{\Phi(t_n)}{u^m(t_n)} \geq \int_0^{t_n} P(s)\mathrm{d}s. \tag{31}$$

Let $n \to \infty$, take the limit of the above equation, because Φ satisfies the condition iii), $I = 0$ is obtained, thus $P = 0$, which contradicts the condition $P \not\equiv 0$, thus the proposition is proved.

4. Discussion on monotonicity

Let $u(0)=U_0$, obviously, when $t=0$, we obtain

$\int_0^t P(t-s)u^m(s)\mathrm{d}s=0,$

So, we can get $\Phi(u(0))=L(0)$ from equation (1), hense $u(0)=\Phi^{-1}(L(0))$.

Theorem 3.1 Let function Φ be continuously differentiable and satisfies the conditions i) - iii), L,P be continuously differentiable and P is non-increasing. When $m>0$, if

$L'(t)+U_0^m P(t)>0$ for all $t\ge 0$,

Then the solution $u(t)$ of equation (1) is strictly increasing on $[0,+\infty)$.

Proof. Since function Φ is continuously differentiable on $[0,+\infty)$, so $u(t)$ is also continuously differentiable on $[0,+\infty)$. Take the derivative of both sides of equation (1) to t, we have

$$\begin{aligned}&\Phi'(u(t))u'(t)\\&=L'(t)+P(0)u^m(t)+\int_0^t P'(t-s)u^m(s)\mathrm{d}s\end{aligned}\tag{32}$$

Let $t=0$ in formula (32), we have

$$\Phi'(u(0))u'(0)=L'(0)+P(0)u^m(0).\tag{33}$$

By our assumption $L'(0)+P(0)u^m(0)>0,$ we obtain

$$\Phi'(u(0))u'(0)>0.\tag{34}$$

Note that Φ increases on $[0,+\infty)$, that is, for any $u>0$, there is $\Phi'(u)>0$, so

$$u'(0)>0.\tag{35}$$

That is, there exists a number $\varepsilon>0$ such that $u'(t)>0$ for all $t\in[0,\varepsilon)$.

Suppose there exists a constant $T>0$ such that $u'(t)>0$ for all $t\in[0,T),$ and $u'(T)=0$. Let $t=T$ in formula (32) we have

$$\begin{aligned}0&=\Phi'(u(T))u'(T)\\&=L'(T)+P(0)u^m(T)+\int_0^T P'(T-s)u^m(s)\mathrm{d}s.\end{aligned}\tag{36}$$

Since $u'(t)>0$ for all $t\in[0,T)$, so $u(t)$ is strictly increasing on $[0,T)$, that is, for any $s\in[0,T)$, there is $u(s)>u(0)>0$. So, when $m>0$, there is $u^m(s)>u^m(0)$, we deduce

$$\begin{aligned}&L'(T)+P(0)u^m(T)+\int_0^T P'(T-s)u^m(0)\mathrm{d}s\\&\le L'(T)+P(0)u^m(T)+\int_0^T P'(T-s)u^m(s)\mathrm{d}s=0,\end{aligned}\tag{37}$$

Hence

$$L'(T)+P(0)u^m(T)+u^m(0)\int_0^T P'(T-s)\mathrm{d}s\le 0.\tag{38}$$

According to formula (38), we have

$$L'(T)+P(0)u^m(T)-u^m(0)\int_0^T P'(T-s)\mathrm{d}(T-s)\le 0\tag{39}$$

Thus

$$L'(t)+P(0)u^m(T)-u^m(0)P(0)+u^m(0)P(T)\le 0 \tag{40}$$

Because $u^m(T)>u^m(0),$ we then observe that

$L'(t)+u^m(0)P(T)\le 0$

From the above equation. note that P is a nonincreasing function, that is, $P(t)\le P(T)$ for any $t\in[0,T)$, we deduce

$$\begin{aligned} L'(t)+U_0^mP(t)&=L'(t)+u^m(0)P(t)\\ &\le L'(t)+u^m(0)P(T)\le 0. \end{aligned} \tag{41}$$

This is contrary to the assumption $L'(t)+U_0^mP(t)>0$, so there is no $T>0$ such that $u'(T)=0$, so $u(t)$ is strictly increasing on $[0,+\infty)$.

Theorem 3.2 Let function Φ be continuously differentiable and satisfies the conditions i) - iii), L,P be continuously differentiable and P is nondecreasing functions. When $m>0$, if

$L'(t)+U_0^mP(t)<0$ for all $t\ge 0$,

Then the solution $u(t)$ of equation (1) strictly decreasing on $[0,+\infty)$.

Note: The proof of Theorem 3.2 is similar to that of Theorem 3.1, so no detailed proof will be made here.

5. Conclusion

On the basis of O. Lipovan's research, this paper considers a more general class of nonlinear convolutional integral equations. By using Arzela – Ascoli theorem, Schauder fixed point theorem and Gronwall inequality, we discuss the existence and uniqueness of the solution of the equation, and give the necessary and sufficient conditions for the boundedness of the nonnegative solution of the equation. Under the given conditions, we prove that the nonnegative solution is monotone. However, no concrete examples have been given for the application of the theorem, which needs further study.

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