The iterative method for the Poisson equation based on domain decomposition

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Abstract. This paper is concerned with 2D Poisson equation. A region-segmentation-based iteration algorithm for the numerical solution of the Poisson equation is presented. The problem can be solved in parallel, where numerical results test our method is effective.

Keywords: Poisson equation; iteration method; domain decomposition.

1. Introduction

Poisson equation is an elliptical partial differential equation, which is widely used in the fields of physics, fluid dynamics and so on. Let Ω be a simply rectangular domain, we consider the following 2D Poisson equation

$$\begin{cases} \dfrac{\partial^2\varphi}{\partial x^2}+\dfrac{\partial^2\varphi}{\partial y^2}=g(x,y), & (x,y)\in\Omega, \\ \varphi(x,y)=p(x,y), & (x,y)\in\partial\Omega. \end{cases} \tag{1}$$

We suppose that the solution $\varphi(x,y)$ and the function $p(x,y)$ are smooth enough. For simplicity, the domain is given as $\Omega=[0,a]\times[0,b]$ and the boundary

$$\partial\Omega=\{(x,y)\,|\,x=0,a,0\le y\le b;\ y=0,b,0\le x\le a\}\ .$$

It is extremely important to study the numerical solution of the Poisson equation. Many scholars have done a lot of work in this field. See the corresponding works. Mittal and Gahlaut[1] gave the second order and fourth order difference formats for the solution of Poisson equation under polar coordinates. Jie Wang et.[3] derived the compact fourth order numerical format. Yin Wang, Jun Zhang[4] developed a sixth order finite difference scheme combined with many numerical techniques. A fourth-order scheme with different grid sizes in different coordinate directions is proposed to solve the 3D Poisson equation by Yongbin Ge[6].

These papers mostly focused on how to construct numerical schemes with high precision. By comparison, higher order accurate discrete methods need more complex procedure. Thus, a big linear system is often generated. It's hard to be computed compared with the system obtained from the low precision format. Therefore, we are interested in how to design the effective algorithm to solve the Poisson equation easily using some high-precision format. Domain decomposition can divide the solution region into serval sub-regions, and then in different sub regions, we can use different numerical methods, so that a complex large-scale system can be decomposed into several small systems to realize parallel computing. Applying the method, many scholars derived the effective algorithms to solve the varies of the partial differential equations. The purpose of this paper is to solve the numerical solution of Poisson equation via an iteration algorithm based on the non-overlapping domain decomposition method. This algorithm only uses the classical five-point and nine-point schemes, but we come true the problem can be computed explicitly while ensuring high accuracy.

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Further, the whole process can be implemented in parallel.

The structure of our paper is as follows. In section II, we discuss the difference schemes including classical five-point scheme and nine-point scheme. In section III, we decompose the domain into some subdomains, thus mesh points are divided into three types: boundary points, interface points and interior points. For each type, we use different numerical schemes and then give the region-segmentation-based iteration algorithm. Finally, in section IV, we give two numerical examples to test our method.

2. Difference Schemes

We first specify the grids. Let the step size $h_x = a/N_1$ in x direction and $h_y = b/N_2$ in y direction, here N_1 and N_2 are interval numbers in x and y directions. For simplicity, we set $h_x = h_y = h$. Then we can get the grid points

$$x_i = ih, i = 0,\cdots, N_1; y_j = jh, j = 0,\cdots, N_2.$$

The notations $\varphi(x_i, y_j), g_{i,j}, p_{i,j}$ express the accurate values of $\varphi(x,y), g(x,y), p(x,y)$ at the mesh point (x_i, y_j).

Making use of Taylor expansion, we have

$$\frac{(\delta_x^2+\delta_y^2)\varphi(x_i,y_j)}{h^2} = \Delta\varphi(x_i,y_j) + \frac{h^2}{12}\left(\frac{\partial^4\varphi}{\partial x^4}+\frac{\partial^4\varphi}{\partial y^4}\right)_{(x_i,y_j)} + O(h^4), \quad \text{where}$$

$$\delta_x^2\varphi(x_i,y_j) = \varphi(x_{i+1},y_j) - 2\varphi(x_i,y_j) + \varphi(x_{i-1},y_j),$$
$$\delta_y^2\varphi(x_i,y_j) = \varphi(x_i,y_{j+1}) - 2\varphi(x_i,y_j) + \varphi(x_i,y_{j-1}).$$

Denote φ_{ij} is the approximate solution, then we can get the classical five-point difference scheme

$$\frac{\varphi_{i+1,j} - 2\varphi_{i,j} + \varphi_{i-1,j}}{h^2} + \frac{\varphi_{i,j+1} - 2\varphi_{i,j} + \varphi_{i,j-1}}{h^2} = g_{i,j}, \tag{2}$$

that is,

$$\varphi_{i+1,j} + \varphi_{i-1,j} - 4\varphi_{i,j} + \varphi_{i,j+1} + \varphi_{i,j-1} = h^2 g_{i,j}. \tag{3}$$

Apparently, the accuracy of scheme (3) is second order. If we discrete the term $\frac{\partial^4 u}{\partial x^4}, \frac{\partial^4 u}{\partial y^4}$, we can get nine-point difference scheme with fourth order accuracy

$$\begin{aligned}&\frac{\varphi_{i+1,j} - 2\varphi_{i,j} + \varphi_{i-1,j}}{h^2} + \frac{\varphi_{i,j+1} - 2\varphi_{i,j} + \varphi_{i,j-1}}{h^2} - \frac{1}{12}[4\varphi_{i,j} - 2(\varphi_{i-1,j} \\ &+\varphi_{i,j-1} + \varphi_{i+1,j} + \varphi_{i,j+1}) + (\varphi_{i-1,j-1} + \varphi_{i+1,j-1} + \varphi_{i-1,j+1} + \varphi_{i+1,j+1})] \\ &= g_{ij} + \frac{h^2}{12}\Delta g_{i,j}. \qquad (4)\end{aligned} \tag{4}$$

In addition, we can get another five-point scheme

$$\varphi_{i-1,j+1} + \varphi_{i+1,j+1} - 4\varphi_{i,j} + \varphi_{i-1,j-1} + \varphi_{i+1,j-1} = 2h^2 g_{i,j}. \tag{5}$$

This scheme has the same order with the scheme (3).

Moreover, some scholars introduced other high order difference schemes. Most of these schemes are implicit including schemes (3) and (5). We can solve (1) by applying these numerical schemes and they all result in a large linear equation system, but it's hard to compute, especially for the high order schemes. So next we will give an iteration algorithm using the idea of non-overlapping domain decomposition. This algorithm is constructed by combining the five-point scheme and the nine-point scheme. It's easy to compute and only if the initial values are set, the whole procedure can be solved explicitly.

3. Iteration Method

Now we begin to express the iterative algorithm. We divide the domain Ω into some subdomains and every subdomain has nine points. Fig. 1 depicts the meshgrid, where "•" points correspond to boundary points, "o" points correspond to interface points and "×" points correspond to interior points.

When i=0 or i=N_1, and j=0 or j=N_2, points are defined as boundary points. We can also specify interface points and interior points. It's convenient to introduce the collection of all grid points by

$$\Gamma = \{(ih, jh): 0 \le i \le N_1, 0 \le j \le N_2\}.$$

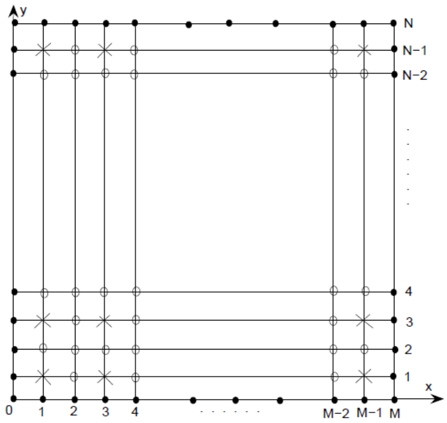

Figure 1 The meshgrid sketch

For simplicity, we assume N_1 and N_2 are even. Denote by Γ_1 the set of boundary points, Γ_2 the set of interface points and Γ_3 the set of interior points.

$$\Gamma_1 = \{(0, jh), (N_1 h, jh), j = 0, \cdots, N_2; (ih, 0), (ih, jN_2), i = 0, \cdots, N_1\},$$

$$\Gamma_2 = \{(2ih, jh), i = 1, \cdots \frac{N_1 - 2}{2}, j = 1, \cdots, N_2 - 1;$$

$$((2i-1)h, 2jh), i = 1, \cdots, \frac{N_1}{2}, j = 1, \cdots, \frac{N_2 - 2}{2}\},$$

$$\Gamma_3 = \{ ((2i-1)h, (2j-1)h), i = 1, \cdots, \frac{N_1}{2}, j = 1, \cdots, \frac{N_2}{2}\}.$$

The region-segmentation-based iteration algorithm can be formulated as follows. First, we give a guessed value at interface points. Then we can calculate the new values at interior points explicitly by using the nine-point scheme (4) in each subdomain. After that, we update the values at interface points by using the five-point scheme (3) and (6). This iteration process can be repeated. For 4×4 mesh as an example, Figure 2 depicts the iteration procedure.

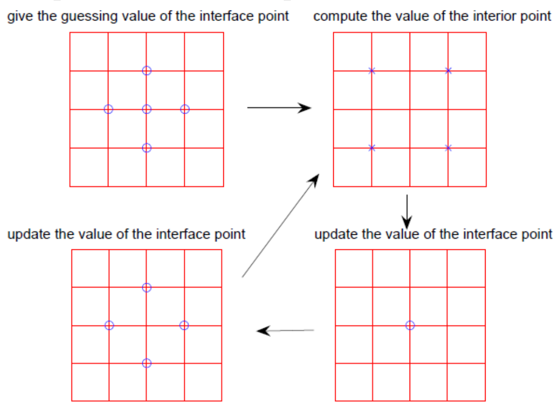

Figure 2 The iteration procedure for 4×4 mesh

Now, we give the iteration algorithm.

3.1. Algorithm:

Let m=0.

(1) For $j=0,\cdots,N_2$, $\varphi_{0,j}^{(m)}=p(0,jh)$, $\varphi_{N_1,j}^{(m)}=p(N_1h,jh)$, and for $i=0,\cdots,N_1$, $\varphi_{i,0}^{(m)}=p(ih,0)$, $\varphi_{i,N_2}^{(m)}=p(ih,N_2h)$.

(2) Set

$$\varphi_{2i,j}^{(m)}=\phi^0(i=1,\cdots,\frac{N_1-2}{2},j=1,\cdots,N_2-1),$$
$$\varphi_{2i-1,2j}^{(m)}=\phi^0(i=1,\cdots,\frac{N_1}{2},j=1,\cdots,\frac{N_2-2}{2}).$$

(3) Compute the values of the interior mesh points $((2i-1)h,(2j-1)h),i=1,\cdots,\frac{N_1}{2},j=1,\cdots,\frac{N_2}{2}$ applying the nine-point scheme (4),

$$(4+\frac{h^2}{3})\varphi_{2i-1,2j-1}^{(m+1)}=(1+\frac{h^2}{6})(\varphi_{2i-2,2j-1}^{(m)}+\varphi_{2i-1,2j-2}^{(m)}+\varphi_{2i,2j-1}^{(m)}+\varphi_{2i-1,2j}^{(m)})$$
$$-\frac{h^2}{12}(\varphi_{2i-2,2j-2}^{(m)}+\varphi_{2i,2j-2}^{(m)}+\varphi_{2i-2,2j}^{(m)}+\varphi_{2i,2j}^{(m)})-h^2g_{2i-1,2j-1}-\frac{h^4}{12}\Delta g_{2i-1,2j-1}.$$

(4) Update the values of the interface mesh point. For mesh points $(2ih,2jh),i=1,\cdots,\frac{N_1-2}{2},j=1,\cdots,\frac{N_2-2}{2}$, we use the five-point scheme (5), i.e.

$$4\varphi_{2i,2j}^{(m+1)}=\varphi_{2i+1,2j+1}^{(m)}+\varphi_{2i-1,2j+1}^{(m)}+\varphi_{2i+1,2j-1}^{(m)}+\varphi_{2i-1,2j-1}^{(m)}-2h^2g_{2i,2j}.$$

For mesh points $(2ih,(2j-1)h),i=1,\cdots,\frac{N_1-2}{2},j=1,\cdots,\frac{N_2}{2}$, and $((2i-1)h,2jh),i=1,\cdots,\frac{N_1}{2},j=1,\cdots,\frac{N_2-2}{2}$, we use the five-point scheme (3), then we get

$$4\varphi_{2i,2j-1}^{(m+1)}=\varphi_{2i+1,2j-1}^{(m)}+\varphi_{2i-1,2j-1}^{(m)}+\varphi_{2i,2j}^{(m)}+\varphi_{2i,2j-2}^{(m)}-h^2g_{2i,2j-1},$$
$$4\varphi_{2i-1,2j}^{(m+1)}=\varphi_{2i,2j}^{(m)}+\varphi_{2i-2,2j}^{(m)}+\varphi_{2i-1,2j+1}^{(m)}+\varphi_{2i-1,2j-1}^{(m)}-h^2g_{2i-1,2j}.$$

(5) Set m=m+1, return to (3).

Repeating the iteration, we can get $\varphi_{i,j}^{(m)},m=1,2,3,\cdots$.

4. Numerical Examples

In section III, we construct an iteration algorithm by the typical difference schemes. We divide the whole linear equation system into subsystems which are solved in iterations. By using the five-point schemes and the nine-point scheme rationally, our method can solve the Poisson equation explicitly. And only if the initial values of the interface points are given, the whole system can be solved in parallel. In this section, we will test our method is effective.

4.1. Example 1

We consider the following problem

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=(x^2+y^2)e^{xy},\quad (x,y)\in\Omega=[0,1]\times[0,1]. \tag{6}$$

The exact solution of the problem (6) is $u(x,y)=e^{xy}$, and the boundary conditions can also be given by it. Setting the maximum norm error

$$E=\max_{0\le i\le M,0\le j\le N}|u_{i,j}-u(x_i,y_j)|.$$

Let m denote the number of iterations. The iteration stopping criterion is $\max_{0\le i\le M, 0\le j\le N} |\varphi_{i,j}^{(m)} - \varphi_{i,j}^{(m-1)}| \le \varepsilon$. In the implementation mentioned above, $\varepsilon = 10^{-8}$.

In Table I, we give the maximum norm error, the iteration number and the iteration time. Compared with the MEGSOR iteration method in [8] for the same sample, we can see the error is 3.6935e-5 when N_1=N_2=128. Apparently, our results are superior to their's.

Table 1. Iteration RESULTS of example 1

Mesh size	*Error*	*Iteration number*	*Iteration time (Seconds)*
N_1=N_2=10	4.1813e-4	117	0.038
N_1=N_2=20	9.7295e-5	429	0.149
N_1=N_2=40	2.3002e-5	1566	1.533
N_1=N_2=80	3.0110e-6	5663	11.411

Next, we regard two mesh sizes with H=2h. The maximum norm errors are denoted as E^h and E^H. We define the convergence order as r with the following form

$$r = \frac{\ln(E^H / E^h)}{\ln 2}.$$

In Table II, we give the corresponding results.

Table 2. convergence order of example 1

Mesh size	*Convergence order*
$h=\frac{1}{20}, H=\frac{1}{10}$	2.1
$h=\frac{1}{40}, H=\frac{1}{20}$	2.1
$h=\frac{1}{80}, H=\frac{1}{40}$	2.9

The error results with different h are shown in Fig. 3, where $\varphi^{(m)}$ is the numerical solution, φ^* is the exact solution. And taking N_1 = 40, we get the images of exact and approximate solutions. They are shown in Fig. 4.

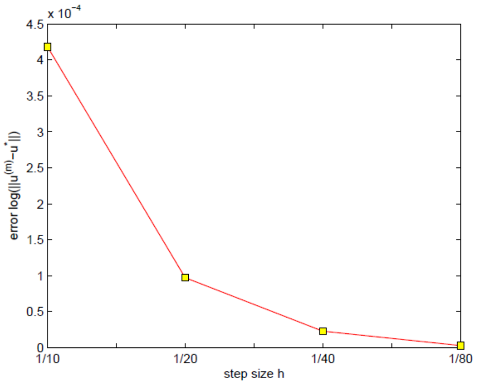

Figure 3. Convergence speed

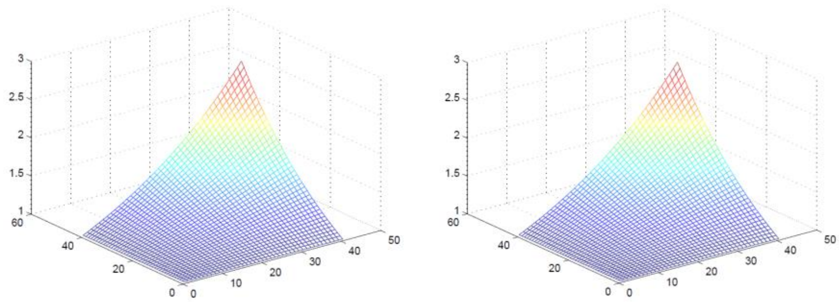

Figure 4. The exact solution(left) and the approximation solution(right)

4.2. Example 2

Next, we consider the problem $\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=-2\pi^2\sin(\pi x)\cos(\pi y), \quad (x,y)\in\Omega=[0,4]\times[0,1]$. This problem has the exact solution with the form of $\varphi(x,y)=\sin(\pi x)\cos(\pi y)$. Since we use the same step size $h_x=h_y$, we set N_1=4N_2. As for Example 1, we also give the maximum norm error, the iteration number, the iteration time and the convergence order. Table III and Table IV give the corresponding results.

Table 3. Iteration RESULTS of example 2

Mesh size	*Error*	*Iteration number*	*Iteration time(Seconds)*
N_1=32,N_2=8	7.7272e-3	31	0.047
N_1=64, N_2=16	1.5358e-3	113	0.078
N_1=128, N_2=32	3.6495e-4	413	0.624
N_1=256, N_2=64	8.9695e-5	1500	7.363

Table 4. Convergence order of example 2

Mesh size	*Convergence order*
$h=\frac{1}{16},H=\frac{1}{8}$	2.3
$h=\frac{1}{32},H=\frac{1}{16}$	2.1
$h=\frac{1}{64},H=\frac{1}{32}$	2.0

Taking N_1=128, N_2=32, we get the images of exact and approximate values. They are shown in Fig.5.

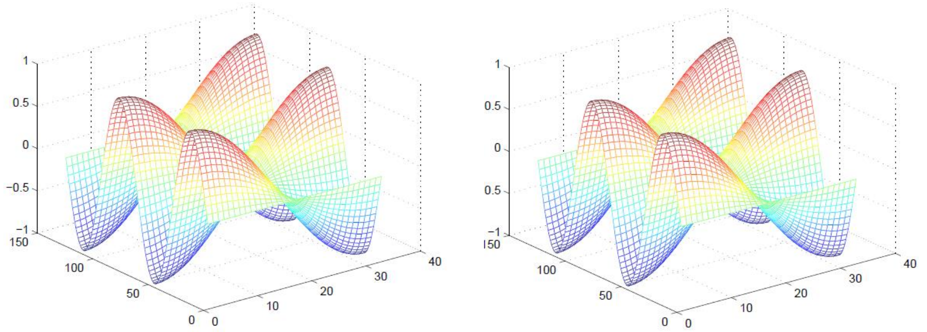

Figure 5. The exact solution(left) and the approximation solution(right)

5. Conclusion

In this paper, we give a new iteration algorithm to solve the Poisson equation only using classical five point and nine point difference schemes. The new method doesn't focus on how to construct high accuracy but complex difference schemes and then how to solve a large algebra system. Our idea is domain decomposition and then the large system becomes small systems that can be calculated in parallel. We decompose Ω into several non-overlapping sub-domains with nine points. Then all mesh points are divided into three sorts: boundary points, interface points and interior points. For different mesh points, we use different numerical schemes which including the five- point and the nine-point schemes. Of course, we can also use other difference schemes in the corresponding sub-domains. Then the whole system can be computed explicitly. From the numerical examples we can see that our algorithm can get highly accurate solution. The method can be done for solving other PDEs.

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