

The Taylor's Theorem and Its Application

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Abstract. Taylor's theorem is an essential concept in calculus. By constructing polynomials, Taylor's formula can simplify calculations by approximating complex functions so that a variety of functions can be analyzed in detail. This paper explores the form and proof of Taylor's theorem based on Peano and Lagrange remainder forms as well as the Polynomial interpolation. Then, examples of the application of Taylor's theorem in mathematics fields, such as limit calculation and high-order derivatives computation, are discussed. Additionally, the applications of Taylor's theorem in other related subjects are examined. Based on the chronological investigation, the Taylor theorem is widely used in physics, engineering, and other fields as a computational tool. The Taylor theorem has gradually been used in machine learning because of the rapid development of computer science in recent years. The purpose of this paper is to summarize the different forms and proofs of Taylor's formula and discuss the development of its application over time.

Keywords: Taylor's Theorem, remainder, Application.

1. Introduction

The study of calculus focuses on the differentiation and integration of functions, and the related concepts and applications. For the study of the properties of a complex function, it is convenient to approximate it with a simple polynomial function. Taylor's formula is one approach of approximating complex functions. Taylor's theorem is a formula that uses the data about a function to describe its value around a given point. If the function satisfies certain conditions in a range, it can be approximated using the Taylor's theorem by constructing an infinite sum of polynomials with the values that contain the derivatives as the coefficients.

This article mainly studies Taylor's theorem. It was created by English mathematician Brook Taylor. Because of the need to resolve complex functions by approximations, Taylor's theorem has been extended and applied in many fields. In 2015, the authors discussed and evaluated different models of thermal energy storage in order to reach a higher energy efficiency [1]. The authors used the Taylor's theorem to calculate the heat. In 2017, Srivastava, Devendra and Jagdev offered an efficient method to analyze the vibration equation for large membranes, a concept studied in science and engineering of the membranes. In 2017, Ding et al. examined the facial expression recognition system that could recognize and classify human facial expression based on analysis in images [3]. They integrated Taylor expansion into the system for better performance. The article by Radim expanded the Taylor Theorem that makes the highest power of derivative take jump discontinuities [4]. The study conducted by Li, Zhu and Cambria discussed the mathematical description of neural tensor networks using the Taylor's Theorem and explored further improvements of the efficiency of the network [5]. In 2021, the researchers evaluated the recurrent neural networks by linking it to Taylor series to improve performance without changing the architecture [6]. According to the study conducted by Navarro, Millwater, et al, the multicomplex Taylor series expansion is introduced as a computational method to improve the evaluation of the sensitivity of phononic metamaterials [7]. In 2022, the study innovatively integrates the Taylor polynomials to the kinetic rate equations, which served as a proof of the concept [8]. In 2021, the researchers constructed a neural network that the number of hidden layers depends on the order of polynomials of the Taylor's theorem and the dimension of input values [9]. In addition, the Cauchy's integral formula is a central idea in complex analysis. It conveys that the function defined in a closed contour is determined by its values on the boundary in the complex plain.

The Section 2 of this article introduces different proof of the Taylor's theorem with various forms of remainder. Then Section 3 introduces some applications of the Taylor's theorem.

2. Proof of the Taylor Theorem

Theorem 2.1 (Taylor's Theorem with Peano's Form of Remainder) If $f(x)$ has n th derivatives at x_0 , then

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + r_n(x), \quad (1)$$

Where the remainder $r_n(x) = o((x - x_0)^n)$.

Proof: For $r_n(x) = f(x) - \sum_{k=0}^n \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k$, it needs to proof $r_n(x) = o((x - x_0)^n)$.

Because $r_n(x_0) = r'_n(x_0) = r''_n(x_0) = \cdots = r^{(n-1)}_n(x_0) = 0$, using the Hospital Rule,

$$\begin{aligned} \lim_{x \rightarrow x_0} \frac{r_n(x)}{(x - x_0)^n} &= \lim_{x \rightarrow x_0} \frac{r'_n(x)}{n(x - x_0)^{n-1}} = \lim_{x \rightarrow x_0} \frac{r''_n(x)}{n(n-1)(x - x_0)^{n-2}} = \cdots = \lim_{x \rightarrow x_0} \frac{r^{(n-1)}_n(x)}{n(n-1)\cdots 2(x - x_0)} \\ &= \frac{1}{n!} \lim_{x \rightarrow x_0} \left[\frac{f^{(n-1)}(x) - f^{(n-1)}(x_0) - f^{(n)}(x_0)(x - x_0)}{x - x_0} \right] = \frac{1}{n!} [f^{(n)}(x_0) - f^{(n)}(x_0)] = 0. \end{aligned} \quad (2)$$

So $r_n(x) = o((x - x_0)^n)$.

Theorem 2.2 (Taylor's Theorem with Lagrange Remainder) If $f(x)$ has n th derivatives for $[a, b]$ and $n+1$ derivatives for (a, b) ,

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + r_n(x), \quad (3)$$

Where the remainder $r_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}$ (ξ is between x and x_0).

Proof: Consider

$$G(t) = f(x) - \sum_{k=0}^n \frac{1}{k!} f^{(k)}(t)(x - t)^k \quad \text{and} \quad H(t) = (x - t)^{n+1}. \quad (4)$$

It needs to prove $G(x_0) = \frac{f^{(n+1)}(\xi)}{(n+1)!} H(x_0)$. Let $x_0 < x$, then $G(t)$ and $H(t)$ are continuous on $[x_0, x]$, differentiable on (x_0, x) , and

$$G'(t) = -\frac{f^{(n+1)}(t)}{n!}(x - t)^n, H'(t) = -(n+1)(x - t)^n. \quad (5)$$

It is obvious that $H'(t)$ is not zero on (x_0, x) , because $G(x) = H(x) = 0$.

Using the Cauchy Mean Value Theorem,

$$\frac{G(x_0)}{H(x_0)} = \frac{G(x) - G(x_0)}{H(x) - H(x_0)} = \frac{G'(\xi)}{H'(\xi)} = \frac{f^{(n+1)}(\xi)}{(n+1)!} \quad \xi \in (x_0, x). \quad (6)$$

Then

$$G(x_0) = \frac{f^{(n+1)}(\xi)}{(n+1)!} H(x_0). \quad (7)$$

Theorem 2.3 If $f(x)$ has n th derivatives on $[a, b]$ and $n+1$ derivatives on (a, b) . The $m+1$ points on $[a, b]$ $x_0, x_1, x_2 \dots x_m$ have known value $f^j(x_i) (i = 0, 1, 2 \dots m, j = 0, 1, 2 \dots n_i - 1; \sum_{i=0}^m n_i = n + 1)$, then for any $x \in [a, b]$,

$$r_n(x) = f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^m (x - x_i)^{n_i}, \quad (8)$$

Where ξ is a number between $x_{\min} = \min(x_0, x_1 \dots x_m, x)$ and $x_{\max} = \max(x_0, x_1 \dots x_m, x)$.

Proof: If x is a given value on $[a, b]$, x is an interpolation node that the remainder is zero, the conclusion is established. If $x \neq x_i (i = 0, 1, 2 \dots m)$, consider the $n+1$ polynomial

$$\omega_{x+1}(t) = \prod_{i=0}^m (t - x_i)^{n_i}, \quad (9)$$

$$\varphi(t) = f(t) - p_n(t) - \frac{\omega_{n+1}(x_i)}{\omega_{n+1}(x)} (f(x) - p_n(x)). \quad (10)$$

The derivative of $\varphi(x)$ at any j th derivative for any $x_i (j = 0, 1 \dots n_i - 1)$ is

$$\varphi^{(j)}(x_i) = f^{(j)}(x_i) - p_n^{(j)}(x_i) - \frac{\omega_{n+1}^{(j)}(x_i)}{\omega_{n+1}(x)} (f(x) - p_n(x)) = 0, \quad (11)$$

$$\varphi(t) = f(t) - p_n(t) - \frac{\omega_{n+1}(x_i)}{\omega_{n+1}(x)} (f(x) - p_n(x)) = 0. \quad (12)$$

So, there are at least $m_0 + 1$ points for $\varphi(x) = 0$, and at least $m_j (j = 1, 2 \dots n + 1)$ points for $\varphi^{(j)}(x) = 0$.

When $j \leq n + 1$, there are $\sum_{k=0}^j m_k - j + 1$ points for $\varphi^{(j)}(t)$ in $[x_{\min}, x_{\max}] \subset [a, b]$.

So when $j = n + 1$,

$$\sum_{k=0}^{n+1} m_k - (n + 1) + 1 = \sum_{k=0}^{n+1} m_k - n = (n + 1) - n = 1. \quad (13)$$

There is at least one point $\xi \in (x_{\min}, x_{\max})$ that makes

$$\varphi^{(n+1)}(\xi) = 0. \quad (14)$$

Because $p_n(t)$ is a n th polynomial, so $p_n^{(n+1)}(t) = 0$, and $\omega^{(n+1)}_{n+1}(t) = (n + 1)!$. It follows

$$0 = \varphi^{(n+1)}(\xi) = f^{(n+1)}(\xi) - \frac{(n+1)!}{\omega_{n+1}(x)} (f(x) - p_n(x)). \quad (15)$$

$$r_n(x) = f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^m (x - x_i)^{n_i}. \quad (16)$$

The interpolation polynomial satisfies the above conditions exists and is unique.

Proof: If $p_n(x)$ and $q_n(x)$ both satisfy the interpolation conditions, consider $p_n(x) - q_n(x)$.

Due to the conditions, it has $\sum_{i=0}^m n_i = n + 1$ roots. But $p_n(x) - q_n(x)$ is a polynomial which power is not exceeding n .

According to the Fundamental Theorem of Algebra,

$$p_n(x) \equiv q_n(x). \quad (17)$$

3. Application

There are several applications of the Taylor Theorem in other mathematical concepts. Some of them are discussed in the following.

When the numerator and denominator of a fraction have the same order, the quantity in the equation can be replaced by Taylor's formula.

Example 3.1: Please find $\lim_{n \rightarrow 0} \frac{\sqrt{4-x} + \sqrt{4+x} - 4}{x^2}$.

Proof: Because the numerator and denominator have the same order, $\sqrt{1-x}$ and $\sqrt{1+x}$ can be replaced by Taylor's formula with Peano's form of remainder to $r(x^2)$.

$$\sqrt{4-x} = 2 + \frac{1}{4}x - \frac{1}{64}x^2 + r_1(x^2). \quad (18)$$

$$\sqrt{4+x} = 2 - \frac{1}{4}x - \frac{1}{64}x^2 + r_2(x^2) \quad (19)$$

Therefore:

$$\begin{aligned} \lim_{n \rightarrow 0} \frac{\sqrt{4-x} + \sqrt{4+x} - 4}{x^2} &= \lim_{n \rightarrow 0} \frac{2 + \frac{1}{4}x - \frac{1}{64}x^2 + r_1(x^2) + 2 - \frac{1}{4}x - \frac{1}{64}x^2 + r_2(x^2) - 4}{x^2} \\ &= \lim_{n \rightarrow 0} \frac{-\frac{1}{32}x^2 + r_1(x^2) + r_2(x^2)}{x^2} = -\frac{1}{32}. \end{aligned} \quad (20)$$

For differentiable functions of infinite order, using Taylor's formula at fixed points will greatly simplify the operation.

Example 3.2 ([10]): For $y = x^2 \sin x$, find $f^{(7)}(0)$.

Proof:

$$y = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n. \quad (21)$$

$$y = x^2 \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \dots \right) = x^3 - \frac{1}{3!}x^5 + \frac{1}{5!}x^7 + \dots. \quad (22)$$

Comparing the variable gives that

$$\frac{f^{(7)}(0)}{7!} x^7 = \frac{1}{5!} x^7. \quad (23)$$

Therefore

$$f^{(7)}(0) = 42. \quad (24)$$

4. Conclusion

Taylor's theorem is a method of approximating functions in calculus. The theorem forms a Taylor polynomial based on the estimation of a differentiable function around a given point. By approximating with polynomials, the characteristics of a function can be expressed in a simpler form, and the computation is also reduced. Therefore, Taylor's theorem is widely used in various subjects. In the calculation and processing of data, the theorem can help analyze and describe functions. Taylor's theorem has inspired more research for diverse investigation. The Taylor's theorem has formed based on Peano Remainder, Lagrange Remainder, and Polynomial Interpolation. There are

mathematical applications such as finding limits and finding higher order derivatives. Considering the various forms of Taylor's theorem allows researchers to apply it in diverse ways. Its application in mathematics also provides unique solutions to a problem. The further study of Taylor's theorem has extensive significance.

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