Schwarz's lemma on the perplex plane

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Abstract. Perplex number is an extension of complex numbers, constituting a commutative ring with zero divisors. In this paper, we primarily investigate the Schwarz lemma on the perplex plane and successfully obtain an estimation of the modulus of differentiable functions in perplex plane. Furthermore, we discuss its potential applications in complex analysis and geometric function theory. This conclusion not only offers a fresh perspective for mathematical research but also holds significant reference value for advancing studies in physics, providing valuable theoretical support for subsequent scientific investigations.

Keyword: Schwarz lemma, Perplex number, Differentiable functions in perplex plane

1. Introduction

The perplex number system has captivated scholars due to its rich applications across various fields. In 1991, A. K. T. Assis explored the utility of perplex numbers in wave equations and generalized algebras, shedding light on their pedagogical significance[1]. Advancing to 2009, Jerry L.R. Chandler emphasized the system's potential in representing electrical particles, introducing a groundbreaking notation for chemistry that eschews traditional symbols[2]. In 2021, Anwane, S. W. revolutionized the understanding of special relativity by articulating its concepts through perplex numbers, promising simplified interpretations and applications in quantum mechanics and nuclear physics[3].

In recent years, the Schwarz lemma has seen significant advancements in the realm of complex variables and geometry. In 2014, Liu Bingyuan presented dual applications of Schwarz lemmas, emphasizing the connection between domain geometries and curvatures[4]. By 2019, Ni Lei bridged earlier Schwarz Lemmata with his own research, offering insights into Kähler manifolds and their curvatures[5]. In 2021, Broder Kyle further expanded the field by introducing two new curvatures, providing a more comprehensive understanding of the Schwarz lemma across different geometric frameworks[6].

The integration of perplex plane and Schwarz lemma studies promises to chart novel territories in the realm of physics. As these mathematical concepts are transposed into physical contexts, they furnish fresh perspectives that challenge traditional paradigms. Such interdisciplinary endeavors stand to invigorate the momentum of physics research, driving innovation and deepening our understanding of the universe's intricate tapestry.

2. Preliminaries

2.1. Perplex numbers

Perplex numbers, denoted as

$$\mathbb{P} := \{c = a + bh : a, b \in \mathbb{R}\},$$

constitute a unique mathematical set distinguished by the hyperbolic unit 'h' satisfying $h^2 = 1$ and $h \neq \pm 1$. This intricate algebraic structure, defined over real components 'a' and 'b', and we call this set of number the perplex numbers, split-complex numbers, duplex numbers and spacetime numbers.

Perplex numbers, introduced in the mid-20th century, owe their inception primarily to Clifford Truesdell. They serve to provide a mathematical framework capable of describing multi-dimensional phenomena, especially within the domain of continuum mechanics.

And we know $\mathbb{C} := \{z = x + iy : x, y \in \mathbb{R}\}$, which is a field. Similar as complex number properties, the operation of addition and multiplication under the set $\mathbb{P}$ would be as following:

$$c_1 + c_2 = (a_1 + a_2) + (b_1 + b_2)h \tag{1}$$

and

$$c_1 c_2 = (a_1 a_2 + b_1 b_2) + (a_1 b_2 + a_2 b_1)h. \tag{2}$$

where $c_1 = (a_1 + b_1 h)$ and $c_2 = (a_2 + b_2 h)$. The zero-unit $'h'$ has the form $(1+h)(1-h) = 0$ since $(1+h)(1-h) = 1 - h + h - h^2$, and $h^2 = 1$, $(1+h)(1-h) = 1 - 1 = 0$ .we can define the real part and perplex part of c as $\mathrm{Re}(c) = a$, $\mathrm{Im}(c) = b$. Now the ring of hyperbolic number has zero-divisors. It also can be written as

$$h_+ := \frac{1+h}{2} \quad \text{and} \quad h_- := \frac{1-h}{2}.$$

So the $\{h_+, h_-\}$ is the idempotent base of $\mathbb{P}$. Furthermore, these zero divisors have the following properties,

$$\begin{aligned} &h_+ \times h_- = 0, \quad h_+ \times h_+ = h_+, \quad h_- \times h_- = h_- \\ &h_+ + h_- = 1, \quad h_+ - h_- = h \end{aligned} \tag{3}$$

So the $\{h_+, h_-\}$ are idempotent zero divisors, they form the whole ring. For such $c = a + bh \in \mathbb{P}$, the idempotent representation would be written as

$$c = c_1 h_+ + c_2 h_-.$$

Where $c_1 = a + b$ and $c_2 = a - b$

Exploring zero-divisors in the mathematical set $\mathbb{P}$ reveals that these entities are exclusively real multiples of h_+ and h_- situated along the lines $a = b$ and $a = -b$. Zero-divisors in $\mathbb{P}$ exist if and only if they can be expressed as $c = ah_+$ or $c = bh_-$, with a and b being nonzero real numbers. The interactions between zero-divisors within $\mathbb{P}$, such as $c_1 = x_1 h_+ + x_2 h_-$ and $c_2 = y_1 h_+ + y_2 h_-$, we have

$$\begin{aligned} c_1 + c_2 &= (x_1 + y_1)h_+ + (x_2 + y_2)h_- \\ c_1 c_2 &= (x_1 y_1)h_- + (x_2 y_2)h_-. \end{aligned} \tag{4}$$

The set of non-negative perplex numbers is

$$\mathbb{P}^+ := \left\{ c_1 = xh_+ + yh_- : x \geq 0, y \geq 0 \right\}, \tag{5}$$

similarly,

$$\mathbb{P}^- := \left\{ c_2 = xh_+ + yh_- : x \leq 0, y \leq 0 \right\}. \tag{6}$$

which is called the set of non-positive hyperbolic numbers. Since such $x, y \in \mathbb{R}$, $c_1 c_2$ is the form of the perplex number, then if $c_1, c_2 \in \mathbb{P}^+ \Rightarrow c_1 c_2 \in \mathbb{P}^+$ .That is, $\mathbb{P}^+$ is closed under the multiplication.

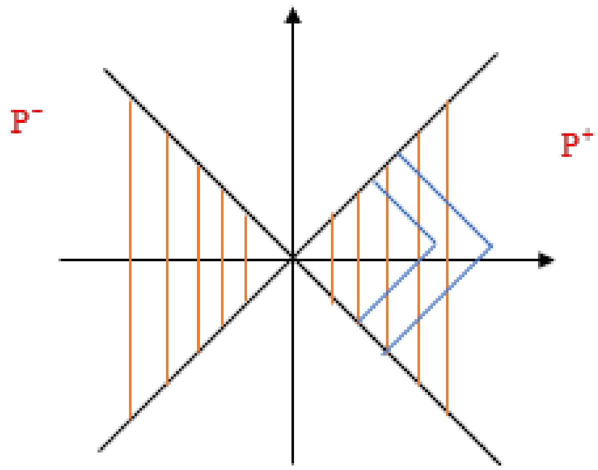

Figure 1. Concept of positive and negative perplex numbers, also the graph of $c, d \in \mathbb{P}$

Consider such $c, d \in \mathbb{P}$, we have

$$c \preceq d \Leftrightarrow d - c \in \mathbb{P}^+$$

which means d $\mathbb{P}$-greater than c, and

$$c \succeq d \Leftrightarrow d - c \in \mathbb{P}^-$$

means d is $\mathbb{P}$ -less than c. Then we can give the following definition[8].

Definition 2.1 For $c, d \in \mathbb{P}$ that $c = x_1 h_+ + y_1 h_-, d = x_2 h_+ + y_2 h_-$, and $c \le d$,there is a closed perplex interval $[c, d]_{\mathbb{P}}$ given by

$$[c, d]_{\mathbb{P}} = \{\delta \in \mathbb{P} : c \preceq \delta \preceq d\}. \tag{7}$$

Trivially, for $c = x_1 h_+ + y_1 h_-, d = x_2 h_+ + y_2 h_-, \delta = \delta_1 h_+ + \delta_2 h_-$,

$$c_i \preceq \delta_i \preceq d_i, i = 1, 2 \quad \Leftrightarrow \quad \delta \in [c, d]_{\mathbb{P}}. \tag{8}$$

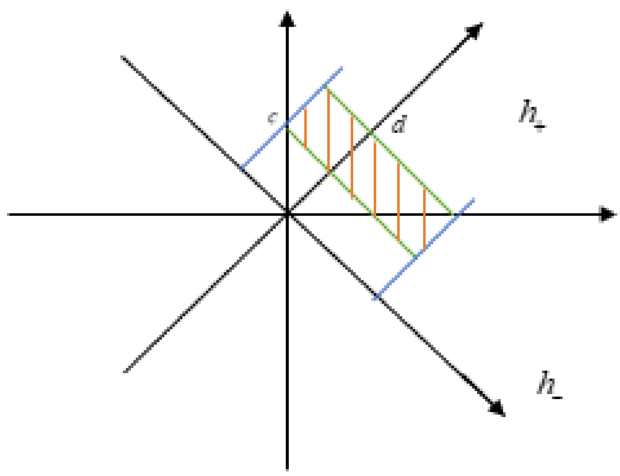

Figure 2. The domain $[c, d]_{\mathbb{P}}$ of the perplex numbers.

Perplex intervals $[c, d]_{\mathbb{P}}$ are defined by the nature of the hyperbolic numbers involved: a non-negative zero divisor perplex number $d - c$ results in a degenerate interval, while an invertible positive hyperbolic number $d - c$ creates a non-degenerate interval, both having a concept of length analogous to real intervals.

A perplex number, denoted as $c = a + bh = xh_+ + yh_-$, can be equivalently represented using either the coordinates $(a, b) \in \mathbb{R}^2$ or $(x, y) \in \mathbb{R}^2$. Additionally, the modulus of hyperbolic numbers can be defined using either the standard base or the idempotent base. Let us see the modulus of the perplex number with idempotent base, we can find it just similar as the complex numbers.

Definition 2.2 A perplex number is given by $c = xh_+ + yh_-$, then positive perplex number such like the form

$$|c|_{\mathbb{P}} = \left|xh_+ + yh_-\right|_{\mathbb{P}} = |x|h_+ + |y|h_- \tag{9}$$

called the perplex modulus of c [7].

Its properties are still similar to the complex number.

$$\begin{aligned} &1. |c|_{\mathbb{P}} = 0 \Leftrightarrow c = 0 \\ &2. |cd|_{\mathbb{P}} = |c|_{\mathbb{P}} |d|_{\mathbb{P}} \\ &3. |c+d|_{\mathbb{P}} \preceq |c|_{\mathbb{P}} + |d|_{\mathbb{P}}\ . \end{aligned} \tag{10}$$

3. Schwarz lemma on perplex plane

3.1. Analysis of Functions of a Perplex Variable

Lemma 1. The $\mathbb{P}$ -valued function f

$f = f_1(c_1)h_+ + f_2(c_2)h_- \Leftrightarrow f$ is $\mathbb{P}$ -differentiable.

Similarly, $f = f_1(c_2)h_+ + f_2(c_1)h_- \Leftrightarrow f$ is $\mathbb{P}$ -differentiable, for such $f_i(c_i), i = 1, 2$ is differentiable.

Using this updated approach to assess differentiability, we can verify that counterparts of certain holomorphic functions in□ exhibit $\mathbb{P}$ -differentiability in $\mathbb{P}$ [10].

After ascertaining the differentiability within $\mathbb{P}$, our subsequent endeavor will be to validate whether the Schwarz lemma maintains its veracity in the perplex plane.

3.2. Schwarz Lemma on the Perplex plane

Now, we can present the formulation of the Schwarz lemma in the context of the perplex plane and endeavor to substantiate its validity therein. Before proceeding with the proof of the Schwarz lemma, it's essential to elaborate on the $\mathbb{P}$-norm. The $\mathbb{P}$-norm is defined on the perplex plane, $\mathbb{P}$, where it provides a measure of the magnitude of perplex numbers. Specifically, for this lemma, we focus our attention on the unit disc defined by the $\mathbb{P}$-norm. Formally, the domain $|c|_{\mathbb{P}}$ and codomain $|f(c)_{\mathbb{P}}|$ of our function are both constrained within this unit disc.

Theorem 1. Let $f : \Delta \to \Delta$ be $\mathbb{P}$ -differentiable with $f(0) = f_1(0)h_+ + f_2(0)h_- = 0$,where $\Delta = \{z \mid |z|_{\mathbb{P}} \leq, z \in \mathbb{P}\}$. Then $|f(c)|_{\mathbb{P}} \leq |c|_{\mathbb{P}}$ for all $c \in \mathbb{D}$.

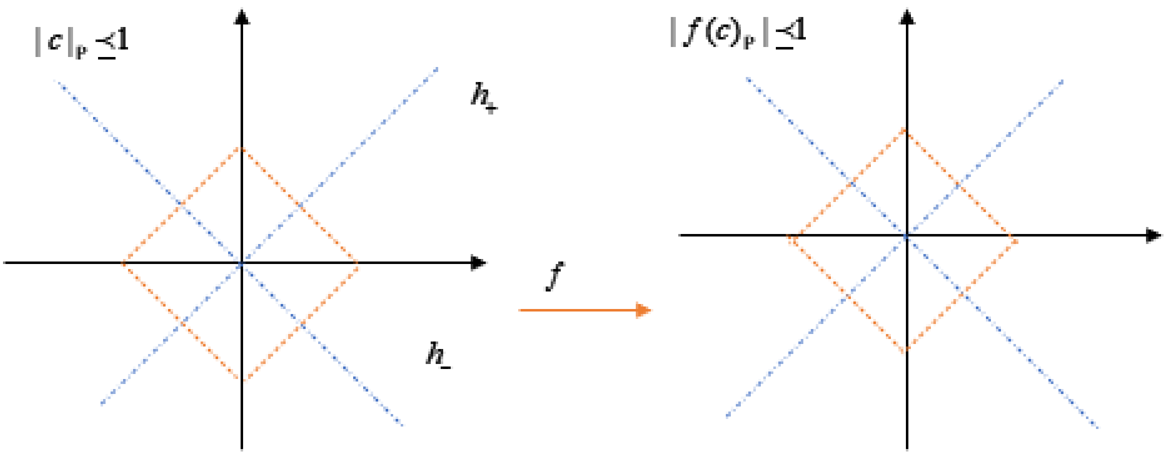

Figure 3. The circle in Δ maps to Δ that $|c|_{\mathbb{P}} \preceq 1 \to |f(c)|_{\mathbb{P}} \preceq 1$ is holomorphic.

Proof.

Let's consider a function $f:\Delta \to \Delta$ that is $\mathbb{P}$-differentiable. Given that $f(c)=f_1(c_1)h_+ + f_2(c_2)h_-$, and since $f(0)=f_1(0)h_+ + f_2(0)h_- = 0$, we deduce that both $f_1(0)$ and $f_2(0)$ are zero.

Using the Taylor Series Expansion, $f_1(c_1)$ and $f_2(c_2)$ in their power series, we have:

$$\begin{aligned} f_1(c_1) &= a_0 + a_1c_1 + a_2c_1^2 + \cdots \\ f_2(c_2) &= b_0 + b_1c_2 + b_2c_2^2 + \cdots. \end{aligned} \tag{11}$$

Given our initial observations, we conclude $a_0 = b_0 = 0$.

The functions $\frac{f_1(c_1)}{c_1}$ and $\frac{f_2(c_2)}{c_2}$ are also differentiable, which is a direct consequence of the above expansions and their term-by-term differentiability.

For the modulus condition $|c_1|=|c_2|=r=1$, given that $|f(c)|_{\mathbb{P}}=|f_1(c_1)|h_+ + |f_2(c_2)|h_- \preceq 1$,it follows (10):

$$|f_1(c_1)|\le 1 \text{ and } |f_2(c_2)|\le 1. \tag{12}$$

Therefore, the modulus of the ratio $\frac{f(c)}{c}$ is [7] :

$$\begin{aligned} \left|\frac{f(c)}{c}\right|_{\mathbb{P}} &= \left|\frac{f_1(c_1)h_+ + f_2(c_2)h_-}{c_1h_+ + c_2h_-}\right|_{\mathbb{P}} \\ &= \left|\frac{f_1(c_1)}{c_1}h_+ + \frac{f_2(c_2)}{c_2}h_-\right|_{\mathbb{P}} \\ &= \left|\frac{f_1(c_1)}{c_1}\right|h_+ + \left|\frac{f_2(c_2)}{c_2}\right|h_- \\ &\preceq \frac{1}{r}h_+ + \frac{1}{r}h_- = \frac{1}{r}. \end{aligned} \tag{13}$$

Taking the limit as $r \to 1$, we deduce:

$$\left|\frac{f(c)}{c}\right|_{\mathbb{P}} \preceq 1 .$$

Which leads us to:

$$|f(c)|_{\mathbb{P}} \preceq |c|_{\mathbb{P}}. \tag{14}$$

This result beautifully mirrors the behavior of holomorphic functions in the complex plane and establishes a bound on the modulus of the function in terms of its argument.

4. Conclusions

In summation, the interdisciplinary study between the perplex plane and the Schwarz lemma offers profound implications for the advancement of mathematical physics. The exploration has paved the way for deeper insights into the structural intricacies of mathematical systems and their potential physical applications. By analyzing the collective findings and advancements across various published works, it becomes evident that the bridge between abstract mathematical concepts and tangible physical phenomena is not only possible but rich in potential. As the field continues to evolve, the integration of these studies holds the promise of unlocking further mysteries and applications, stimulating the momentum for future research and discoveries.

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