Lagrange's mean value theorem in hyperbolic number plane

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Abstract. Hyperbolic number is a generalization of real number, it is a commutative ring of zero factor, and has been widely used in many fields such as mathematics, physics and engineering. In this paper, we first review the basic concepts and properties of hyperbolic numbers, and then study the Lagrange's mean value theorem on the hyperbolic numbers plane, and get a more general theorem. This not only provides impetus for the development of hyperbolic analysis, but also further injects energy into the development of applied mathematics.

Keywords: Hyperbolic numbers, Lagrange's mean value theorem, Functions of a hyperbolic variable.

1. Introduction

The research background of hyperbolic numbers can be traced back to the end of the 19 th century and the beginning of the 20 th century. Scholars at that time noted that although complex numbers can well represent many mathematical phenomena, they do not fully cover all complex situations. Thus, hyperbolic numbers came into being. In the 20th century, hyperbolic numbers became a tool to describe the Lorentz transformation of special relativity and became a key research object.

Sobczyk [1] discussed the properties and algebraic structure of the hyperbolic numbers plane, in particular the multiplication and division rules for hyperbolic numbers. Representations, operations on hyperbolic numbers, and their geometric and algebraic applications are also obtained. Atkins [2] studied the geometric and mathematical properties of hyperbolic numbers and provides a way to describe hyperbolic affine iterations. Djavvat [3] obtained how hyperbolic numbers can be used to describe and analyze invariants in split complex plane. For more details, please see[4-8].

As we know, Lagrange's mean value theorem is one of the fundamental theorems of differential calculus, is an important tool of modern mathematics, and is widely used in physics and engineering. Therefore, based on the above study, we have deeply explored the Lagrange median theorem in the hyperbolic plane, and this study will not only provide theoretical support for the field of mathematics, but also provide a strong research impetus for further development in the field of physics. The in-depth exploration of this study is expected to provide us with more comprehensive and accurate tools in solving physical problems in the hyperbolic plane, and to promote greater progress in our understanding of nature.

2. Preliminaries

2.1. The preparatory content of hyperbolic numbers

Hyperbolic numbers are commutative rings with zero factors, represented as follows:

$$\mathbb{R}^{1,1} = \{\xi = x + yj : x , y \in \mathbb{R}, j^2 = 1\} \quad . \tag{1}$$

Where the hyperbolic unit j satisfies $j^2 = 1$ and $j \neq \pm 1$. The hyperbolic number is the Clifford algebra $\mathcal{C}\ell_{0,1}$ generating 1 and j as an algebra over $\mathbb{R}$. $\mathbb{R}^{1,1}$ is similar to $\mathbb{C}$, however, it has a different algebraic structure, in which addition and multiplication satisfy the following rule:

$$\zeta_1 + \zeta_2 = (x_1 + y_1 j) + (x_2 + y_2 j) = (x_1 + x_2) + (y_1 + y_2) j \tag{2}$$

$$\begin{aligned} \zeta_1 \zeta_2 &= (x_1 + y_1 j)(x_2 + y_2 j) = x_1 x_2 + x_1 y_2 j + y_1 j x_2 + y_1 j y_2 j \\ &= (x_1 x_2 + y_1 y_2) + (x_1 y_2 + y_1 x_2) j, \end{aligned} \tag{3}$$

where $\zeta_1 = x_1 + y_1 j$ and $\zeta_2 = x_2 + y_2 j$. For any $\zeta = x + yj \in \mathbb{R}^{1,1}$, where the real part of ζ is $\mathrm{Re}(\zeta) = x$, the hyperbolic part of ζ is $\mathrm{Im}(\zeta) = y$. The conjugate hyperbolic number of ζ is expressed as $\overline{\zeta} = x - yj$. The hyperbolic number ring has zero divisor. Two special zero divisors of the hyperbolic number is expressed as follows:

$$j_+ := \frac{1+j}{2} \text{ and } j_- := \frac{1-j}{2} \tag{4}$$

$$j_+ j_- = \left(\frac{1+j}{2}\right)\left(\frac{1-j}{2}\right) = \frac{1 - j + j - j^2}{4} = 0, \tag{5}$$

where $\{j_+, j_-\}$ are called the two bases of the hyperbolic number.

It is easy to obtain some calculation results of zero factor:

$$(j_+)^2 = \left(\frac{1+j}{2}\right)\left(\frac{1+j}{2}\right) = \frac{1 + j + j + j^2}{4} = \frac{1+j}{2} = j_+ \tag{6}$$

$$(j_-)^2 = \left(\frac{1-j}{2}\right)\left(\frac{1-j}{2}\right) = \frac{1 - j - j + j^2}{4} = \frac{1-j}{2} = j_- \tag{7}$$

$$j_+ + j_- = \frac{1+j}{2} + \frac{1-j}{2} = 1 \text{ and } j_+ - j_- = \frac{1+j}{2} - \frac{1-j}{2} = j. \tag{8}$$

Decomposition of zero divisors:

$$\zeta = x + yj = (x+y)\frac{1+j}{2} + (x-y)\frac{1-j}{2} := u j_+ + v j_-. \tag{9}$$

Obviously, the zero divisors in set $\mathbb{R}^{1,1}$ are real multiples of j_+ and j_-. Because multiples of j_+ and j_- lie on the $y = x$ and $y = -x$ lines. Therefore, ζ is a zero divisor if and only if $\zeta = x j_+$ or $\zeta = y j_-$, where $x, y \in \mathbb{R}\backslash\{0\}$.

In addition, for $\zeta_1 = u_1 j_+ + v_1 j_-$ and $\zeta_2 = u_2 j_+ + v_2 j_-$, then

$$\zeta_1 + \zeta_2 = (u_1 + u_2) j_+ + (v_1 + v_2) j_- \tag{10}$$

$$\zeta_1 \zeta_2 = u_1 u_2 (j_+)^2 + u_1 v_2 j_+ j_- + v_1 u_2 j_- j_+ + v_1 v_2 (j_-)^2 = u_1 u_2 j_+ + v_1 v_2 j_-. \tag{11}$$

We introduce partial order:

$$\mathbb{R}_+^{1,1} := \{\zeta = u j_+ + v j_- : u \geq 0, v \geq 0\}, \tag{12}$$

which is called the set of nonnegative hyperbolic numbers,

$$\mathbb{R}_-^{1,1} := \{\zeta = u j_+ + v j_- : u \leq 0, v \leq 0\}, \tag{13}$$

which is called the set of nonpositive hyperbolic numbers.

It is not difficult to find that if $\zeta_1, \zeta_2 \in \mathbb{R}_+^{1,1}$, then $\zeta_1\zeta_2 \in \mathbb{R}_+^{1,1}$. Therefore, $\mathbb{R}_+^{1,1}$ is closed by multiplication. So, for any $\zeta_1, \zeta_2 \in \mathbb{R}^{1,1}$, then

$$\zeta_1 \preceq \zeta_2 \text{ if and only if } \zeta_2 - \zeta_1 \in \mathbb{R}_+^{1,1}, \tag{14}$$

and we say that ζ_2 is $\mathbb{R}^{1,1}$—greater than ζ_1, or

$$\zeta_2 \preceq \zeta_1 \text{ if and only if } \zeta_1 - \zeta_2 \in \mathbb{R}_+^{1,1}. \tag{15}$$

And we say that ζ_2 is $\mathbb{R}^{1,1}$—less than ζ_1.

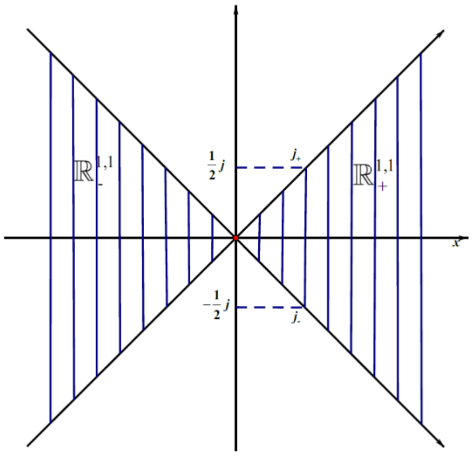

Figure 1. The nonnegative hyperbolic numbers and nonpositive hyperbolic numbers

In addition, $\zeta \in \mathbb{R}_+^{1,1}$ is equivalent to $\zeta \succeq 0$, $\zeta \in \mathbb{R}_-^{1,1}$ is equivalent to $\zeta \preceq 0$. $\zeta_1 = u_1 j_+ + v_1 j_-$ and $\zeta_2 = u_2 j_+ + v_2 j_-$, then $\zeta_1 \preceq \zeta_2$, if $u_1 \le u_2$ and $v_1 \le v_2$. Moreover, relation $\preceq$ is self-reciprocal, transitive, and antisymmetric, so relation $\preceq$ represents the partial order in $\mathbb{R}^{1,1}$. We can see this order of relationships in Figure 2.

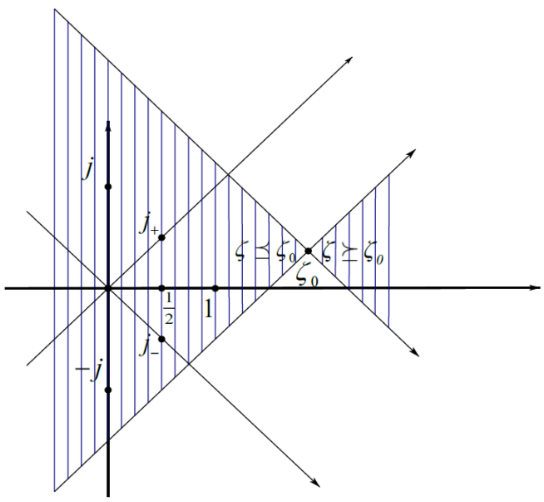

Figure 2. The partial order in $\mathbb{R}^{1,1}$

Definition 1. [9, Definition 2.6.1] When $\zeta_1 = u_1 j_+ + v_1 j_-$ and $\zeta_2 = u_2 j_+ + v_2 j_-$ make $\zeta_1 \prec \zeta_2$ in $\mathbb{R}^{1,1}$, the closed hyperbolic interval $[\zeta_1, \zeta_2]_{\mathbb{R}^{1,1}}$ is defined as

$$[\zeta_1, \zeta_2]_{\mathbb{R}^{1,1}} = \{\zeta \in \mathbb{R}^{1,1} : \zeta_1 \preccurlyeq \zeta \preccurlyeq \zeta_2\}, \tag{16}$$

equivalent to: $\zeta = uj_+ + vj_- \in [\zeta_1, \zeta_2]_{\mathbb{R}^{1,1}}$ if and only if

$$u_1 \le u \le u_2 \ \text{ and } \ v_1 \le v \le v_2. \tag{17}$$

2.2. Functions of a hyperbolic variable

The functions of a hyperbolic variable is defined as follows:

$$f : U \subseteq \mathbb{R}^{1,1} \to \mathbb{R}^{1,1}. \tag{18}$$

These functions have already been investigated in many places. Here U is the open subset of $\mathbb{R}^{1,1}$. These functions can be written as $f(z) = f_1(x,y) + jf_2(x,y)$.

Suppose $f_1, f_2 \in C^1(U)$, we define the limit of the difference quotient:

$$\lim_{\substack{h\to 0\\ h\notin L}} \frac{f(z_0+h)-f(z_0)}{h}, \tag{19}$$

where h cannot be a zero divisor, and we have the following theorem.

Theorem 1. [10, Corollary 2.7] A hyperbolic variable function f is $\mathcal{C}\ell_{0,1}$-differentiable if and only if

$$f = f_1(u)\, j_+ + f_2(v)\, j_-, \tag{20}$$

where $f_1(u)$ and $f_2(v)$ are differentiable as real valued functions of one variable respectively.

3. Results

Theorem 2.(hyperbolic Lagrange's mean value theorem) Suppose that f is continuous in the closed hyperbolic interval $[\delta,\omega]_{\mathbb{R}^{1,1}}$, and Suppose also that f is $\mathcal{C}\ell_{0,1}$-differentiable at each point of an open hyperbolic interval $(\delta,\omega)_{\mathbb{R}^{1,1}}$.Then there exist a point $\zeta \in (\delta,\omega)_{\mathbb{R}^{1,1}}$ such that

$$f(\omega) - f(\delta) = f'(\xi)(\omega - \delta). \tag{21}$$

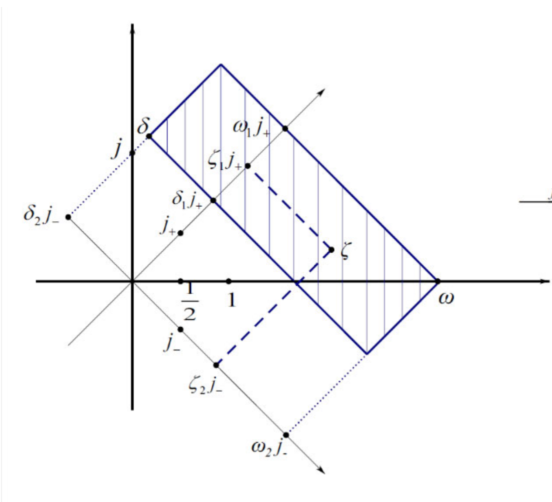

Figure 3. The shaded part is a hyperbolic interval $[\delta,\omega]_{\mathbb{R}^{1,1}}$

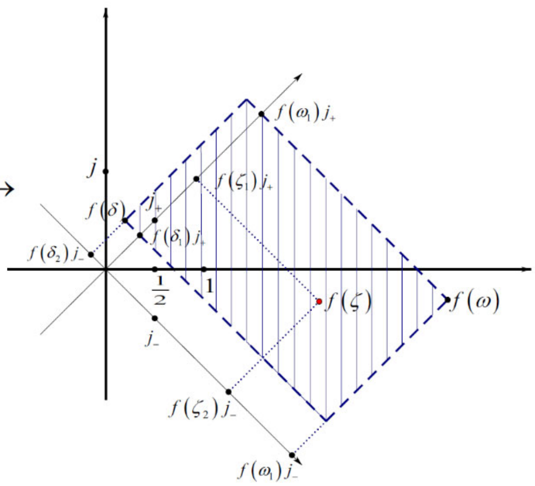

Figure 4. The result of Lagrange's mean value theorem

Proof : Let

$$\delta = \delta_1 j_+ + \delta_2 j_- \tag{22}$$

$$\omega = \omega_1 j_+ + \omega_2 j_-. \tag{23}$$

Since f is $\mathcal{C}\ell_{0,1}$-differentiable, so we have

$$f(\delta)=f_1(\delta_1)j_+ + f_2(\delta_2)j_- \tag{24}$$

$$f(\omega)=f_1(\omega_1)j_+ + f_2(\omega_2)j_-. \tag{25}$$

Combining (24) with (25), we have

$$f(\omega)-f(\delta)=\left[f_1(\omega_1)-f_1(\delta_1)\right]j_+ + \left[f_2(\omega_2)-f_2(\delta_2)\right]j_-. \tag{26}$$

According to the mean value theorem in mathematical analysis, we can get the result that

$$f(\omega)-f(\delta)=f_1'(\xi_1)(\omega_1-\delta_1)j_+ + f_2'(\xi_2)(\omega_2-\delta_2)j_-, \tag{27}$$

where $\xi_1\in[\delta_1,\omega_1],\xi_2\in[\delta_2,\omega_2]$.

It is easy to obtain

$$f(\omega)-f(\delta)=f_1'(\xi_1)\omega_1 j_+ + f_2'(\xi_2)\omega_2 j_- - \left[f_1'(\xi_1)\delta_1 j_+ + f_2'(\xi_2)\delta_2 j_-\right]. \tag{28}$$

On the other hand, by the property of the zero factor, we know

$$\begin{cases} j_+j_+ = j_+ \\ j_-j_- = j_- \\ j_+j_- = 0. \end{cases} \tag{29}$$

Thus

$$f(\omega)-f(\delta)=\left[f_1'(\xi_1)j_+ + f_2'(\xi_2)j_-\right](\omega_1 j_+ + \omega_2 j_-)-\left[f_1'(\xi_1)j_+ + f_2'(\xi_2)j_-\right](\delta_1 j_+ + \delta_2 j_-). \tag{30}$$

Extracting the common factor from (30), we have

$$\begin{aligned} f(\omega)-f(\delta)&=\left[f_1'(\xi_1)j_+ + f_2'(\xi_2)j_-\right](\omega-\delta) \\ &=f'(\xi)(\omega-\delta), \end{aligned} \tag{31}$$

where $\zeta=\zeta_1 j_+ + \zeta_2 j_-$.Thus there exist a point $\zeta\in(\delta,\omega)_{\mathbb{R}^{1,1}}$, such that

$$f(\omega)-f(\delta)=f'(\xi)(\omega-\delta). \tag{32}$$

4. Conclusions

The application background of hyperbolic complex function is very wide, and it has its unique use in many scientific fields. The main innovation of this paper is that we successfully establish the Lagrange mean value theorem on the hyperbolic plane by using the basic concepts and properties of hyperbolic numbers. The value of this work is that it not only provides a more perfect theoretical basis for hyperbolic analysis, but also provides a useful tool for the study of physical science and other fields.

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