

Cauchy's Criterion for Non-negative Hyperbolic Series

Xin Chen^{1, #, *}, Yuxin Li^{2, #}, Zhuolin Zhu^{3, #}

¹College of sciences, Shantou University, Shantou, China, 515821

²Department of mathematics, Lancaster University, Lancaster, United Kingdom, LA14YW

³College of sciences, Shanghai University, Shanghai, China, 200444

*Corresponding author: 22xchen4@stu.edu.cn

#These authors contributed equally.

Abstract. Hyperbolic number plane constitutes a significant area of algebraic research, with their properties finding wide-ranging applications across various fields. Meanwhile, Cauchy's criterion, as an effective tool for determining series convergence, has also been extensively utilized in the study of hyperbolic numbers within mathematical analysis and practical problem-solving, offering valuable insights and practical applications. The paper conducted an in-depth exploration of the non-negative hyperbolic plane, applying the Comparison test and further investigating the Cauchy's criterion on the non-negative hyperbolic plane. Through the application of the comparison test, the paper have arrived at an important conclusion: when considering two numbers, w_n and z_n , in the non-negative hyperbolic complex plane, and $z_n \prec w_n$, the convergence of $\sum w_n$ implies the convergence of $\sum z_n$, and divergence of $\sum w_n$ implies the divergence of $\sum z_n$. Furthermore, through a detailed analysis of Cauchy's criterion, the paper also discovered a key finding: when a certain number r in the non-negative hyperbolic plane satisfies $r = \lim_{n \rightarrow \infty} \sqrt[n]{z_n}$, if $r < 1$, the corresponding infinite series $\sum z_n$ converges, whereas if $r > 1$, the series diverges. The paper not only pushes the frontiers of mathematical theory but also unlocks fresh avenues for interdisciplinary exploration. By delving into uncharted territories, it expands our understanding of mathematical principles and their applicability across diverse fields. Moreover, it fortifies mathematics' central role in addressing real-world challenges, providing essential tools and insights. As a cornerstone for future research, this work paves the way for innovative mathematical investigations and their practical applications, fostering a deeper symbiosis between theory and the real world.

Keywords: Hyperbolic numbers, Hyperbolic-valued norm, Cauchy's criterion.

1. Introduction

Hyperbolic numbers which form a commutative ring with zero divisors serve as an extension of two real numbers. In algebraic research, the structure and properties of hyperbolic numbers have become a cutting-edge topic. In 2016, Gargoubi and Kossentini[1] discovered that hyperbolic numbers are the unique natural extension of real numbers in 2-D Archimedean $\{f\}$ algebras. In 2021, Soykan and Taşdemir[2] further introduced generalized hyperbolic Jacobi Stahl-Lucas numbers to explore their properties and their relationship with hyperbolic Jacobi Stahl-Lucas numbers. This exploration covers Binet formulas, generating functions, summing formulas, and more, providing novel insights into mathematical structures. Additionally, hyperbolic numbers have applications in interdisciplinary fields, such as Sergey.V. Petoukhov's research[3] in the field of biology, which delved deeply into the application of multidimensional hyperbolic numbers. It is evident that the study of hyperbolic numbers holds profound value and significance.

Cauchy's criterion, which serves as an effective method for determining the convergence of series and frequently finds application in both mathematical analysis and practical problem-solving, can play a key role in the study of hyperbolic numbers. For instance, in 2018, Takeshi Fukao and colleagues [4] utilized Cauchy's criterion to study nonlinear diffusion equations defined over

unbounded domains. In 2019, Sagir and Erdogan[5] employed Cauchy's criterion to establish the existence and uniqueness of solutions for initial boundary value problems within the context of non-Newtonian calculus, focusing on function sequences and series. Furthermore, in 2022, Jun Kawabe and Naoki Yamada[6] applied the Cauchy's criterion to develop a convergence theory for measurable functions within a measure space.

Based on the above researches, this paper deeply studies the Cauchy's criterion in non-negative hyperbolic plane. This research not only broadens the scope of mathematical theory, but also provides new directions and possibilities for mathematical research, allowing mathematicians to more fully understand hyperbolic geometry and related mathematical structures. This helps mathematicians better understand the connections between different fields, laying the foundation for solving a wider range of mathematical problems. In addition, this research also provides new opportunities for interdisciplinary research, as the hyperbolic plane and the Cauchy's criterion have practical applications in biology, physics, computer science and other fields. This reinforces the role of mathematics in solving practical problems, providing important foundations and insights for future mathematical research and applications.

2. Preliminaries

2.1. Hyperbolic numbers \mathbb{G}

The algebra of hyperbolic numbers,

$$\mathbb{G} = \{z = a + bi \mid a, b \in \mathbb{R}, i^2 = 1\}$$

is the the Clifford algebra[7] $Cl_{1,0}$ generated as an algebra over \mathbb{R} by 1 and i . Despite the similarities between \mathbb{G} and \mathbb{C} , they possess distinct algebraic structures[8]. Specifically, \mathbb{G} has zero divisors[9]:

$$(1+i)(1-i) = 1+i-i-i^2 = 1-1 = 0 \quad (1)$$

We can define the following:

$$e = \frac{1+i}{2} \quad \text{and} \quad e^+ = \frac{1-i}{2} \quad (2)$$

Furthermore, e and e^+ satisfy the following properties:

$$(1) \text{ The product of } e \text{ and } e^+ \text{ is } 0: ee^+ = \frac{1+i}{2} \cdot \frac{1-i}{2} = \frac{1+i-i-i^2}{4} = \frac{1-1}{4} = 0$$

$$(2) \text{ The square of } e \text{ is equal to itself: } e^2 = ee = \frac{1+i}{2} \cdot \frac{1+i}{2} = \frac{1+i+i+i^2}{4} = \frac{2+2i}{4} = \frac{1+i}{2} = e$$

$$(3) \text{ The square of } e^+ \text{ is equal to itself: } e^{+2} = e^+e^+ = \frac{1-i}{2} \cdot \frac{1-i}{2} = \frac{1-i-i+i^2}{4} = \frac{2-2i}{4} = \frac{1-i}{2} = e^+$$

$$(4) \text{ The sum of } e \text{ and } e^+ \text{ is } 1: e + e^+ = \frac{1+i}{2} + \frac{1-i}{2} = \frac{1+i+1-i}{2} = \frac{2}{2} = 1$$

$$(5) \text{ The value of } e \text{ minus } e^+ \text{ is } i: e - e^+ = \frac{1+i}{2} - \frac{1-i}{2} = \frac{1+i-1+i}{2} = \frac{2i}{2} = i$$

Besides that, we can also obtain a useful foundation for \mathbb{G} :

$$z = a + bi = a\left(\frac{1+i}{2} + \frac{1-i}{2}\right) + b\left(\frac{1+i}{2} - \frac{1-i}{2}\right) = (a+b)e + (a-b)e^+ = ue + ve^+ \quad , \quad (3)$$

where $u = a+b, v = a-b$.

Therefore, in these new coordinates, multiplication simplifies to:

$$\begin{aligned}
 (u_1e + v_1e^+)(u_2e + v_2e^+) &= u_1u_2e^2 + u_1v_2ee^+ + v_1u_2e^+e + v_1v_2e^{+2} \\
 &= u_1u_2\left(\frac{1+i}{2}\right)^2 + u_1v_2\frac{1+i}{2}\cdot\frac{1-i}{2} + v_1u_2\frac{1-i}{2}\cdot\frac{1+i}{2} + v_1v_2\left(\frac{1-i}{2}\right)^2 \quad (4) \\
 &= u_1u_2\frac{1+i}{2} + 0 + 0 + v_1v_2\frac{1-i}{2} = u_1u_2e + v_1v_2e^+
 \end{aligned}$$

The inverse is defined as follows :

$$\begin{aligned}
 z^{-1} &= \frac{1}{a+bi} = \frac{a-bi}{(a+bi)\cdot(a-bi)} = \frac{a-bi}{a^2-b^2i^2} = \frac{2a-2bi}{2(a+b)\cdot(a-b)} \\
 &= \frac{(1+i+1-i)\cdot a - (1+i-1+i)\cdot b}{2(a+b)\cdot(a-b)} = \frac{(1+i)\cdot(a-b) + (1-i)\cdot(a+b)}{2(a+b)\cdot(a-b)} \quad (5) \\
 &= \frac{1+i}{2(a+b)} + \frac{1-i}{2(a-b)} = \frac{1}{a+b}\cdot\frac{1+i}{2} + \frac{1}{a-b}\cdot\frac{1-i}{2} \\
 &= u^{-1}e + v^{-1}e^+
 \end{aligned}$$

Inverting one of the elements is also suitable:

$$z_+^{-1} = u^{-1}e + ve^+ \quad \text{and} \quad z_-^{-1} = ue + v^{-1}e^+. \quad (6)$$

2.2. Partial order and hyperbolic-valued norm

The group of non-negative hyperbolic number is defined by

$$\mathbb{G}^+ = \{z = ue + ve^+ \mid u, v \geq 0\}.$$

The new coordinate system is shown in Figure 1.

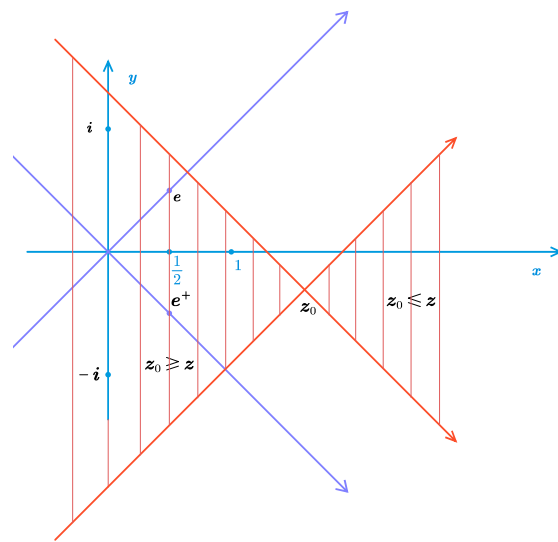
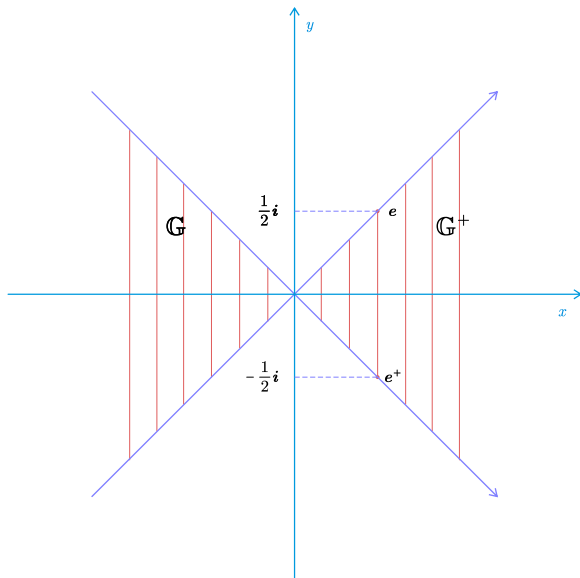


Figure 1:The Positive And Negative Hyperbolic Numbers **Figure 2:**A Partial Order On \mathbb{G}

And the set is closed under hyperbolic multiplication. As a result of this fact, we can define the following relation on \mathbb{G} : given $z_1 = u_1e + v_1e^+, z_2 = u_2e + v_2e^+ \in \mathbb{G}$, if $z_2 - z_1 \in \mathbb{G}^+$, we write

$$z_1 \succeq z_2 \quad (\text{when } u_1 \geq u_2 \text{ and } v_1 \geq v_2) \quad \text{or} \quad z_1 \preceq z_2 \quad (\text{when } u_1 \leq u_2 \text{ and } v_1 \leq v_2)$$

As shown in Figure 2. The relation is symmetric, anti-symmetric and reflexive, concluding then that \preceq is a partial order.

The function $|\cdot|_{\mathbb{G}} : \mathbb{G} \rightarrow \mathbb{G}^+$ defined by

$$|z|_{\mathbb{G}} = |ue + ve^+|_{\mathbb{G}} := |u|e + |v|e^+ \in \mathbb{G}^+ \tag{7}$$

is called a hyperbolic-valued norm.

Lemma1[10,Theorem 10.1.1]Cauchy criteria

The series $\sum_{n=1}^{\infty} z_n$ converges if and only if the sequence of its partial sums is a \mathbb{D} -Cauchy sequence, *i.e.*, for any positive hyperbolic number ε there exists a natural number $N = N(\varepsilon)$ such that

$$\left| \sum_{k=1}^{\infty} z_{N+k} \right|_{\mathbb{D}} \prec \varepsilon$$

for any natural m.

3. Results

Theorem3.1 (Comparison test for non-negative hyperbolic series) Given a non-negative hyperbolic series $\sum_{n=1}^{\infty} z_n$, if there exists a non-negative hyperbolic series $\sum_{n=1}^{\infty} w_n$ such that $z_n \preceq \delta w_n$ holds on $n = n^*, n^{*+1}, n^{*+2} \dots$ for some $n^* \in \mathbb{N}$. Then:

(1) When $\sum_{n=1}^{\infty} w_n$ converges, $\sum_{n=1}^{\infty} z_n$ also converges.

(2) When $\sum_{n=1}^{\infty} z_n$ diverges, $\sum_{n=1}^{\infty} w_n$ also diverges.

Proof:

(1) Since $\sum_{n=1}^{\infty} w_n$ converges, according to [7, Theorem 10.1.1] :

$$\forall \varepsilon \succ 0, \exists N > 0, \forall P > 0, |w_{N+1} + w_{N+2} + \dots + w_{N+P}|_{\mathbb{G}} = \left| \sum_{k=1}^P w_{N+k} \right|_{\mathbb{G}} \prec \varepsilon$$

Since $z_n \preceq \delta w_n$, we have $z_{N+1} + z_{N+2} + \dots + z_{N+P} \preceq \delta(w_{N+1} + w_{N+2} + \dots + w_{N+P})$.

Then $|z_{N+1} + z_{N+2} + \dots + z_{N+P}|_{\mathbb{G}} \preceq \delta |w_{N+1} + w_{N+2} + \dots + w_{N+P}|_{\mathbb{G}}$.

Then $\left| \sum_{k=1}^P z_{N+k} \right|_{\mathbb{G}} \preceq \delta \left| \sum_{k=1}^P w_{N+k} \right|_{\mathbb{G}} \prec \delta \varepsilon$. According to [7, Theorem 10.1.1]: $\sum_{n=1}^{\infty} z_n$ converges.

(2) $\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} z_{1,n} e + \sum_{n=1}^{\infty} z_{2,n} e^+$, $\sum_{n=1}^{\infty} w_n = \sum_{n=1}^{\infty} w_{1,n} e + \sum_{n=1}^{\infty} w_{2,n} e^+$

$\sum_{n=1}^{\infty} z_n$ diverges. Divide into the following three cases.

① $\sum_{n=1}^{\infty} z_{1,n}$ diverges, $\sum_{n=1}^{\infty} z_{2,n}$ converges:

Since $\sum_{n=1}^{\infty} z_{1,n}$ diverges and $z_n \preceq \delta w_n$, we have $\sum_{n=1}^{\infty} w_{1,n}$ diverges. Then $\sum_{n=1}^{\infty} w_n$ diverges

② $\sum_{n=1}^{\infty} z_{1,n}$ converges, $\sum_{n=1}^{\infty} z_{2,n}$ diverges:

Since $\sum_{n=1}^{\infty} z_{2,n}$ diverges and $z_n \preceq \delta w_n$, we have $\sum_{n=1}^{\infty} w_{2,n}$ diverges. Then $\sum_{n=1}^{\infty} w_n$ diverges

③ $\sum_{n=1}^{\infty} z_{1,n}$ diverges, $\sum_{n=1}^{\infty} z_{2,n}$ diverges:

Since $\sum_{n=1}^{\infty} z_{1,n}$ diverges, $\sum_{n=1}^{\infty} z_{2,n}$ diverges, $z_n \preceq \delta w_n$, then $\sum_{n=1}^{\infty} w_{1,n}$ diverges, $\sum_{n=1}^{\infty} w_{2,n}$ diverges.

We have $\sum_{n=1}^{\infty} w_n$ diverges

In conclusion, when $\sum_{n=1}^{\infty} z_n$ diverges, $\sum_{n=1}^{\infty} w_n$ also diverges.

Theorem 3.2 (Cauchy's convergence test for non-negative hyperbolic series)

Given a non-negative hyperbolic series $\sum_{n=1}^{\infty} z_n$, if there exists a non-negative hyperbolic number r

such that $r = \lim_{n \rightarrow \infty} \sqrt[n]{z_n}$, we have:

(1) When $r = r_1 e + r_2 e^+$, $r_1 \prec 1, r_2 \prec 1$, $\sum_{n=1}^{\infty} z_n$ converges. As shown in Figure 3.

(2) When $r = r_1 e + r_2 e^+$, $r_1 \prec 1, r_2 \succ 1$, $\sum_{n=1}^{\infty} z_n$ diverges. As shown in Figure 4.

(3) When $r = r_1 e + r_2 e^+$, $r_1 \succ 1, r_2 \prec 1$, $\sum_{n=1}^{\infty} z_n$ diverges. As shown in Figure 5.

(4) When $r = r_1 e + r_2 e^+$, $r_1 \succ 1, r_2 \succ 1$, $\sum_{n=1}^{\infty} z_n$ diverges. As shown in Figure 6.

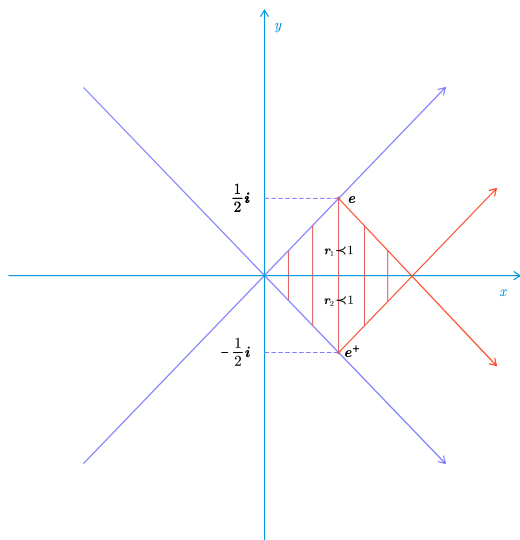


Figure 3: $r_1 \prec 1, r_2 \prec 1, \sum_{n=1}^{\infty} z_n$ converges

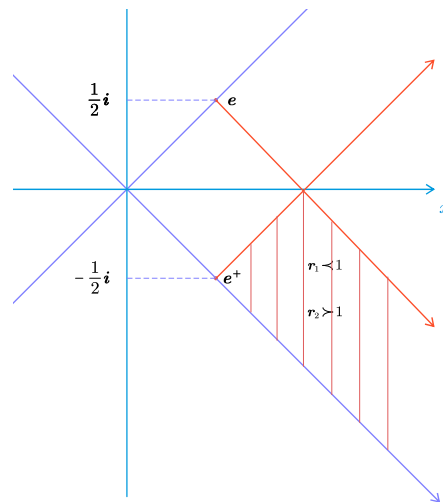


Figure 4: $r_1 \prec 1, r_2 \succ 1, \sum_{n=1}^{\infty} z_n$ diverges.

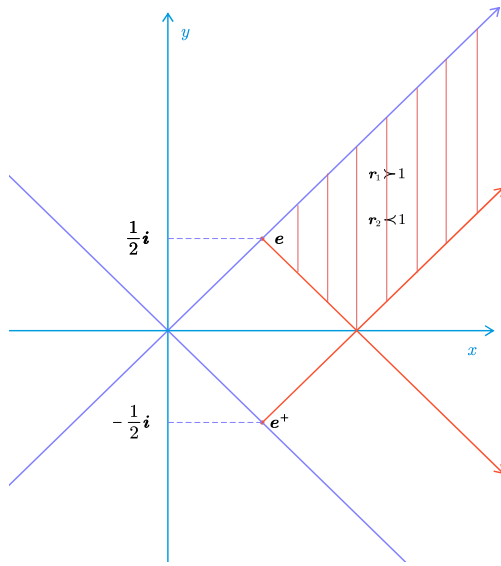


Figure 5: $r_1 > 1, r_2 < 1, \sum_{n=1}^{\infty} z_n$ diverges.

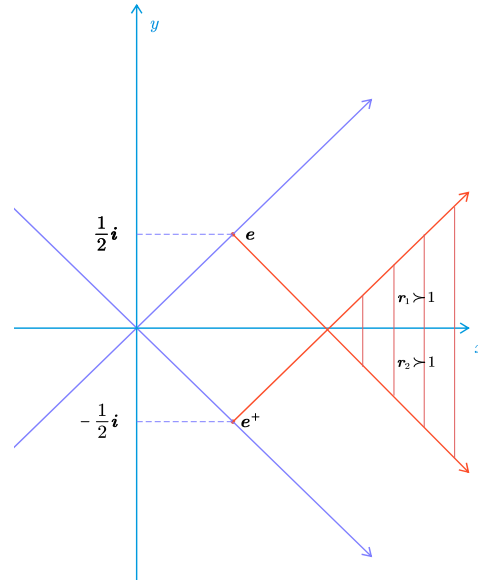


Figure 6: $r_1 > 1, r_2 > 1, \sum_{n=1}^{\infty} z_n$ diverges.

Proof:

$$r = \lim_{n \rightarrow \infty} \sqrt[n]{z_n} = \lim_{n \rightarrow \infty} (\sqrt[n]{z_{1,n}} e + \sqrt[n]{z_{2,n}} e^+) = (\lim_{n \rightarrow \infty} \sqrt[n]{z_{1,n}}) e + (\lim_{n \rightarrow \infty} \sqrt[n]{z_{2,n}}) e^+ = r_1 e + r_2 e^+ . \text{Since } r_1 < 1, r_2 < 1$$

,then $\sum_{n=1}^{\infty} z_{1,n}$ converges, $\sum_{n=1}^{\infty} z_{2,n}$ converges. Then, $\sum z_n$ converges.

$$r = \lim_{n \rightarrow \infty} \sqrt[n]{z_n} = \lim_{n \rightarrow \infty} (\sqrt[n]{z_{1,n}} e + \sqrt[n]{z_{2,n}} e^+) = (\lim_{n \rightarrow \infty} \sqrt[n]{z_{1,n}}) e + (\lim_{n \rightarrow \infty} \sqrt[n]{z_{2,n}}) e^+ = r_1 e + r_2 e^+ \quad . \text{Since}$$

$r_1 < 1, r_2 > 1$,We have $\sum_{n=1}^{\infty} z_{1,n}$ converges, $\sum_{n=1}^{\infty} z_{2,n}$ diverges. Since $\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} z_{1,n} e + \sum_{n=1}^{\infty} z_{2,n} e^+$,then $\sum z_n$ diverges.

$$r = \lim_{n \rightarrow \infty} \sqrt[n]{z_n} = \lim_{n \rightarrow \infty} (\sqrt[n]{z_{1,n}} e + \sqrt[n]{z_{2,n}} e^+) = (\lim_{n \rightarrow \infty} \sqrt[n]{z_{1,n}}) e + (\lim_{n \rightarrow \infty} \sqrt[n]{z_{2,n}}) e^+ = r_1 e + r_2 e^+ \quad . \text{Since}$$

$r_1 > 1, r_2 < 1$,we have $\sum_{n=1}^{\infty} z_{1,n}$ diverges, $\sum_{n=1}^{\infty} z_{2,n}$ converges. Since $\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} z_{1,n} e + \sum_{n=1}^{\infty} z_{2,n} e^+$,then $\sum z_n$ diverges.

$$r = \lim_{n \rightarrow \infty} \sqrt[n]{z_n} = \lim_{n \rightarrow \infty} (\sqrt[n]{z_{1,n}} e + \sqrt[n]{z_{2,n}} e^+) = (\lim_{n \rightarrow \infty} \sqrt[n]{z_{1,n}}) e + (\lim_{n \rightarrow \infty} \sqrt[n]{z_{2,n}}) e^+ = r_1 e + r_2 e^+ \quad . \text{Since}$$

$r_1 > 1, r_2 > 1$,we have $\sum_{n=1}^{\infty} z_{1,n}$ diverges, $\sum_{n=1}^{\infty} z_{2,n}$ diverges. Since $\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} z_{1,n} e + \sum_{n=1}^{\infty} z_{2,n} e^+$,then $\sum z_n$ diverges.

4. Conclusions

In this paper, we conducted an in-depth exploration of the non-negative hyperbolic plane using the Comparison test. Building upon this foundation, we embarked on research concerning the Cauchy's criterion on the non-negative hyperbolic plane. By applying the Comparison test, we have arrived at an important conclusion. Specifically, when considering two numbers which are w_n and z_n in the non-

negative hyperbolic plane that w_n less than z_n , if $\sum_{n=1}^{\infty} w_n$ converges, then $\sum_{n=1}^{\infty} z_n$ must also converge.

Conversely, if $\sum_{n=1}^{\infty} z_n$ diverges, the $\sum_{n=1}^{\infty} w_n$ will also diverge. This conclusion provides us with a convenient method to determine the convergence of sequences. Then, through a meticulous analysis of the mathematical properties associated with the Cauchy's criterion, we also successfully derived a crucial conclusion: when a certain number r on the non-negative hyperbolic plane satisfies the condition $r = \lim_{n \rightarrow \infty} \sqrt[n]{z_n}$, it leads to a significant distinction. When r is less than 1, the corresponding infinite series $\sum_{n=1}^{\infty} z_n$ converges; Conversely, when r more than 1, the series $\sum_{n=1}^{\infty} z_n$ diverges. This finding not only broadens our understanding of the Cauchy's criterion method within the context of the non-negative hyperbolic plane but also provides substantial support and guidance for related research and applications in the field of mathematics.

References

- [1] Gargoubi.H, Kossentini.S. Algebra Structure on Hyperbolic Numbers. in Advances in Applied Clifford Algebras [J], vol. 26, pp. 1211-1233, 2016.
- [2] Soykan.Y, Taşdemir.E. A study on hyperbolic numbers with generalized Jacobsthal numbers components. International Journal of Nonlinear Analysis and Applications [J], vol. 13, no. 2, pp. 1965-1981,2021.
- [3] Sergey.V.Petoukhov. Modeling inherited physiological structures based on hyperbolic numbers. Biosystems[J], vol. 199, pp. 104285, 2021.
- [4] Fukao.T, Kurima.S, Yokota.T. Nonlinear Diffusion Equations as Asymptotic Limits of Cahn-Hilliard Systems on Unbounded Domains via Cauchy's Criterion. Mathematical Methods in the Applied Sciences[J], vol. 41, no. 3, pp. 951-969, 2018.
- [5] Saoir.B, Erdo Gean.F. On the Function Sequences and Series in the Non-Newtonian Calculus. Journal of Science & Arts[J], vol. 19, no. 4, pp. 915-936, 2019.
- [6] Kawabe.J, Yamada.N. The completeness and separability of function spaces in nonadditive measure theory[J]. Fuzzy Sets and Systems, vol. 466, 2022, p. 108409, 2022, ISSN:0165-0114,
- [7] Franchini.S, Vassallo.G, Sorbello.F. A brief introduction to Clifford algebra[J]. Palermo (Italy): Università degli studi di Palermo, 2010.
- [8] Chartier.P, Hairer.E, Vilmart.G. Algebraic structures of B-series[J]. Foundations of Computational Mathematics, 2010, 10: 407-427.
- [9] Tellez-Sanchez.G.Y, Bory-Reyes.J. Extensions of the shannon entropy and the chaos game algorithm to hyperbolic numbers plane[J]. Fractals, 2021, 29(01): 2150013.
- [10] Luna-Elizarrarás, M. Elena, Michael Shapiro, Daniele C. Struppa, Adrian Vajiac. Bicomplex Holomorphic Functions: The Algebra, Geometry and Analysis of Bicomplex Numbers[M]. Birkhäuser, 2015, ISBN: 978-3-319-24866-0.