Convergence Properties of the Perplex Series

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Abstract. Perplex numbers is a number system similar to the complex numbers. The perplex numbers is a commutative ring with zero divisors. The purpose of this article is to determine convergence properties of the perplex series and to give the geometric series in $\mathbb{P}$. Also, we analyze the convergence properties of the geometric perplex series. The properties we show will provide a theoretical basis for its application. The geometric perplex series will be paid more attention in the future and may be used widely in physics.

Keywords: Perplex numbers, Convergence, Geometric perplex series.

1. Introduction

The perplex numbers has abundant applications. It's necessary to analyze its basic properties and discover its fundamental theorems of algebra for them. In 2009, Robert D. Poodiack and Kevin J. Leclair review the basic properties in $\mathbb{P}$, and then prove three fundamental theorems of algebra for the perplex numbers [1]. The perplex numbers has many great properties, such as hyperbolic algebra, which can be used in physics. In 2018, Ronni Geraldo Gomes de Amorim applies the results of the perplex numbers for the study of Lorentz Transformations and the wave equation [2]. The perplex numbers not only can solve wave equation, but also can characterize bounded orbits in dynamical systems. In 2019, there are four mathematicians investigating the analogs of these sets for dynamical systems over the perplex numbers [3].

The geometric series has really rich applications in many fields. Such as physics, communications, engineering, medicine, bioinformatics, chemistry and so on. The geometric series is a model to fit the data. In 2005, Simone Fattorini discovers a simple method to fit geometric series. It is also proposed as a method to discriminate between relict and equilibrial models [4]. In 2009, Chinnaraji Annamalai discovers a generalized theorem, which is a generic digital computing-geometric series model with abundant uses. It will make a difference in science and engineering fields [5]. It is vital to know ways of the computation of the geometric series. There are many other methods for the computation of it [6]-[8]. Convergence is an important property of the geometric series. In 1976, Roger W. Brockett discovers a method to solve controlled differential equations which are analytic and linear. It can be expanded in a Volterra series provided there is no finite escape time. The Volterra kernels are computed in terms of the geometric series. If the series isn't convergent, it's hard to solve equations [9].

The perplex numbers has many great theorems of algebra. What's more, the geometric perplex series has great convergence properties and it keeps some properties of geometric series. These make the geometric perplex series especially useful in solving problems of physics.

2. The basic fundamental of perplex numbers

2.1. The algebra properties of perplex numbers.

The perplex numbers are similar to the complex numbers. The form of the perplex numbers is:

$$\{c = a + bh \mid a, b \in \mathbb{R}, h^2 = 1\},$$

We define $'a'$ is real part of c, and $'b'$ is referred to as hyperbolic part. We also define a perplex conjugate $\overline{c} = a - bh$.

Let $c_1 = a_1 + b_1$ and $c_2 = a_2 + b_2 h$, we suppose arithmetic operations on them:

$$c_1 + c_2 = (a_1 + a_2) + (b_1 + b_2)h, \tag{1}$$

$$c_1 - c_2 = (a_1 - a_2) + (b_1 - b_2)h, \tag{2}$$

$$c_1 c_2 = (a_1 a_2 + b_1 b_2) + (a_2 b_1 + a_1 b_2)h, \tag{3}$$

$$\frac{c_1}{c_2} = \frac{c_1\overline{c_1}}{c_2\overline{c_2}} = \frac{(a_1 + b_1 h)(a_1 - b_1 h)}{(a_2 + b_2 h)(a_2 - b_2 h)} = \frac{a_1 a_2 - b_1 b_2 + (a_2 b_1 - a_1 b_2)h}{a^2 - b^2}. \tag{4}$$

We can know from (3) $c\overline{c} = a^2 - b^2$, it can be positive, negative or zero. When $a = \pm b$, c is called a zero divisor. That's made the perplex numbers different from the complex numbers, it is a commutative ring. Since $h^2 = 1$, $(h+1)(h-1) = 0$, we can get another zero divisors $h_+ = \frac{1+h}{2}$ and $h_- = \frac{1-h}{2}$. $\{h_+, h_-\}$ is called the idempotent base of $\mathbb{P}$.

It exemplifies one of the most special properties of the set of perplex numbers. As the idempotent base of perplex numbers, $\{h_+, h_-\}$ has some properties:

$$h_+ \cdot h_- = (\frac{1+h}{2}) \cdot (\frac{1-h}{2}) = 0, \tag{5}$$

$$h_+ \cdot h_+ = (\frac{1+h}{2}) \cdot (\frac{1+h}{2}) = \frac{1+h}{2} = h_+, \tag{6}$$

$$h_- \cdot h_- = (\frac{1-h}{2}) \cdot (\frac{1-h}{2}) = \frac{1-h}{2} = h_-, \tag{7}$$

$$h_+ + h_- = 1, \tag{8}$$

$$h_+ - h_- = h. \tag{9}$$

For any $c = a + bh \in \mathbb{P}$, we can decompose c as:

$$c = (a+b)(\frac{1+h}{2}) + (a-b)(\frac{1-h}{2}) = (a+b) \cdot h_+ + (a-b)h_- = c_1 \cdot h_+ + c_2 \cdot h_-.$$

The advantage of using the idempotent base of perplex number in all algebraic operations is as following:

$$c_1 \cdot c_2 = a_1 b_1 h_+ + a_2 b_2 h_-. \tag{10}$$

Proof: For any

$$c_1 = a_1 h_+ + b_1 h_-, c_2 = a_2 h_+ + b_2 h_-.$$

According to (5)-(7)

$$\begin{aligned} c_1 c_2 &= (a_1 h_+ + b_1 h_-)(a_2 h_+ + b_2 h_-) \\ &= a_1 a_2 h_+ h_+ + a_1 b_2 h_+ h_- + b_1 a_2 h_+ h_- + b_1 b_2 h_- h_- \\ &= a_1 a_2 h_+ + b_1 b_2 h_-. \end{aligned} \tag{11}$$

The multiplication of perplex numbers is still perplex numbers.

Let we see the definition of the perplex modulus, $c = a + bh$ to be.

$$|c|_{\mathbb{P}} = \sqrt{|c\bar{c}|} = \sqrt{|a^2 - b^2|} \tag{12}$$

Using $\{h_+, h_-\}$, $|c|_{\mathbb{P}}$ can be.

$$|c|_{\mathbb{P}} = |c_1 \cdot h_+ + c_2 \cdot h_-| = |c_1| \cdot h_+ + |c_2| \cdot h_-. \tag{13}$$

$|c|_{\mathbb{P}}$ has some properties:

$$\begin{gathered} |c|_{\mathbb{P}} = 0 \Leftrightarrow c = 0, \\ |cd|_{\mathbb{P}} = |c|_{\mathbb{P}} |d|_{\mathbb{P}}, \\ |c + d|_{\mathbb{P}} \preceq |c|_{\mathbb{P}} + |d|_{\mathbb{P}} \end{gathered} \tag{14}$$

Thus, we can define the set of positive numbers is:

$\mathbb{P}^+ =: \{c = c_1 \cdot h_+ + c_2 \cdot h_-; c_1, c_2 \geq 0\}$,

We will call its elements “positive numbers”, similarly,

$\mathbb{P}^- =: \{c = c_1 \cdot h_+ + c_2 \cdot h_-; c_1, c_2 \leq 0\}$.

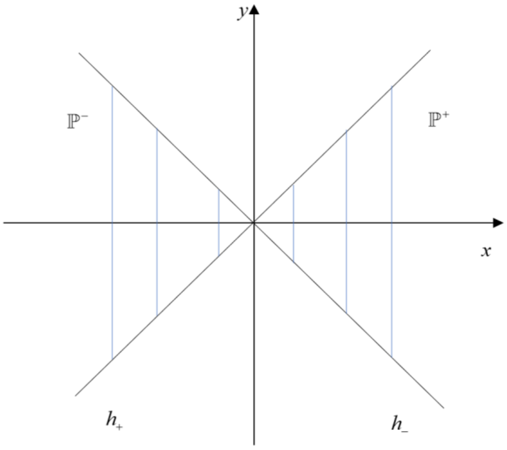

Figure 1: The positive and negative perplex number.

The perplex numbers is different from the complex numbers, the perplex numbers can compare with each other. We next define the order of the perplex numbers.

Definition 2.1 Let $c, d \in \mathbb{P}$, $c \preceq d$, equals to $d - c \in \mathbb{P}^+$, just $c_1 \leq d_1$ and $c_2 \leq d_2$. If $c \succeq d$, $d - c \in \mathbb{P}^-$.

We can define interval in $\mathbb{P}$:

$$[c, d]_{\mathbb{P}} = \{\delta \in \mathbb{P} : c \preceq \delta \preceq d\}, \tag{15}$$

$$\delta = \delta_1 \cdot h_+ + \delta_2 \cdot h_-, c_i \leq \delta_i \leq d_i, i = 1, 2.$$

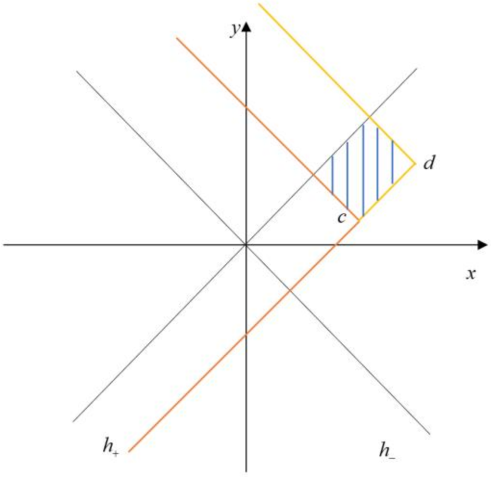

Figure 2: The domain $[c,d]_{\mathbb{P}}$ of the perplex numbers.

3. Results

3.1. The properties of the series of perplex numbers.

Definition 3.1 A sequence of $\mathbb{P}$ numbers $\{C_n\}_{n\in N}$ converge to a perplex number c_0, if for every $\varepsilon \in \mathbb{P}^+$, there exists an $N \in □$ such that whenever $n \geq N$, it follows that $\left|c_n - c_0\right|_{\mathbb{P}} \prec \varepsilon$. We say c_0 is the limit of the sequence which we write as $\lim\limits_{n\to\infty} c_n = c_0$.

If the sequence of $\mathbb{P}$ numbers is convergent, the partial sum of the sequence is also convergent, we can get $\lim\limits_{n\to\infty} S_n = S$, according to the decompose of S:

$$S = S_1 \cdot h_+ + S_2 \cdot h_-,$$

We have

$$\lim_{n\to\infty} S_n = S = S_1 h_+ + S_2 h_- .$$

We discover the limit of perplex series still keeps the addition, subtraction and scalar multiplication operations.

Theorem 3.1 If $\sum\limits_{n=1}^{\infty} c_n$ is convergent, and $\lim\limits_{n\to\infty}\sum\limits_{n=1}^{\infty} c_n = \alpha$. $\sum\limits_{n=1}^{\infty} d_n$ is also convergent and $\lim\limits_{n\to\infty}\sum\limits_{n=1}^{\infty} d_n = \beta$, we can get $\lim\limits_{n\to\infty}\sum\limits_{n=1}^{\infty}(c_n \pm d_n) = \alpha \pm \beta$. $(c_n, \alpha \in \mathbb{P}, d_n, \beta \in \mathbb{P})$

Proof: For any $c_k \in \mathbb{P}$, $d_k \in \mathbb{P}$, we can analyze from the decompose of the perplex numbers,

$$c_n = c_1 h_+ + c_2 h_-, \quad d_n = d_1 h_+ + d_2 h_- .$$

So

$$\sum_{n=1}^{\infty} c_n + d_n = \sum_{n=1}^{\infty}(c_{1n}h_+ + c_{2n}h_-) + (d_{1n}h_+ + d_{2n}h_-) = \sum_{n=1}^{\infty}(c_{1n} + d_{1n})h_+ + \sum_{n=1}^{\infty}(c_{2n} + d_{2n})h_- .$$

According to

$$\lim_{n\to\infty}\sum_{n=1}^{\infty} c_n = \lim_{n\to\infty}\sum_{n=1}^{\infty}(c_{1n}h_+ + c_{2n}h_-) = \alpha = \alpha_1 h_+ + \alpha_2 h_-,$$

$$\lim_{n\to\infty}\sum_{n=1}^{\infty} d_n = \lim_{n\to\infty}\sum_{n=1}^{\infty}(d_{1n}h_+ + d_{2n}h_-) = \beta = \beta_1 h_+ + \beta_2 h_- .$$

Thus

$$\lim_{n\to\infty}\sum_{n=1}^{\infty}(c_n + d_n) = \lim_{n\to\infty}(\sum_{n=1}^{\infty}(c_{1n} + d_{1n})h_+ + \sum_{n=1}^{\infty}(c_{1n} + d_{1n})h_+) = (\alpha_1 + \beta_1)h_+ + (\alpha_2 + \beta_2) = \alpha + \beta .$$

Theorem 3.2 If $\sum_{n=1}^{\infty} c_n$ is convergent, and $c_n \in \mathbb{P}$, $a \in \mathbb{R}$, we can get $\sum_{n=1}^{\infty} ac_n$ is also convergent and $\sum_{n=1}^{\infty} ac_n = a\sum_{n=1}^{\infty} c_n$.

Proof: For any $c_k \in \mathbb{P}$, we can know from the decompose of the perplex numbers,

$$c_k = c_{1k}h_+ + c_{2k}h_- .$$

So

$$\sum_{k=1}^{n} c_k = \sum_{k=1}^{n}(c_{1k}h_+ + c_{2k}h_-) = \sum_{k=1}^{n} c_{1k}h_+ + \sum_{k=1}^{n} c_{2k}h_- .$$

Since

$$\sum_{k=1}^{n} c_k = S_k .$$

The partial sum of c_k is convergent, so we have:

$$\lim_{k\to\infty} S_k = S = S_1 h_+ + S_2 h_- ,$$

$$\lim_{k\to\infty}\sum_{k=1}^{n} ac_k = \lim_{k\to\infty} aS_k = aS = aS_1 h_+ + aS_2 h_- .$$

So $\sum_{n=1}^{\infty} ac_n$ is also convergent and $\sum_{n=1}^{\infty} ac_n = a\sum_{n=1}^{\infty} c_n$.

3.2. Analysis of the convergence of the geometric perplex series

The perplex numbers also has the geometric series, we call it the geometric perplex series, it has the following form:

$$\sum_{n=1}^{\infty} a\delta^n = a + a\delta + a\delta^2 + \cdots + a\delta^n + \cdots, a = a_1 h_+ + a_2 h_-, \delta = \delta_1 h_+ + \delta_2 h_- .$$

If we have the form $a = a_1 h_+ + a_2 h_-$, $(a \neq 0\cdot h_+ + 0\cdot h_-)$, $\delta = \delta_1 h_+ + \delta_2 h_-$, $(a_1, a_2, \delta_1, \delta_2 \in \mathbb{R})$

$$a\delta^n = (a_1 h_+ + a_2 h_-)(\delta_1^n h_+ + \delta_2^n h_-) .$$

According to (10),

$$a\delta^n = a_1\delta_1^n h_+ + a_2\delta_2^n h_- .$$

when $|\delta_1| < 1, |\delta_2| < 1$,

$$\lim_{n\to\infty} a_1 \frac{1-\delta_1^n}{1-\delta_1} = a_1 \frac{1}{1-\delta_1}, \lim_{n\to\infty} a_2 \frac{1-\delta_2^n}{1-\delta_2} = a_2 \frac{1}{1-\delta_2} .$$

So, the geometric perplex series is convergent, and $\lim_{n\to\infty} S_n = a_1 \frac{1}{1-\delta_1} h_+ + a_2 \frac{1}{1-\delta_2} h_- = \frac{a}{1-\delta}$.

Otherwise, the geometric perplex series isn't convergent.

The geometric perplex series in perplex plane is algebra ring without zero divisors, the complex structure of algebra makes it impossible to perform numerical experiments.

4. Conclusions

In this paper, we review the basic properties of the perplex numbers, and we define the geometric perplex series, then analyze the convergence properties of it. The geometric perplex series we discuss is a common series in the perplex plane. The exploration paves the way for a deeper understanding of the structural complexity of mathematical systems and their potential physical applications. It really has a wide use in physics. In the future, more and more people study the properties of perplex numbers, and it will have a wider application in all aspects.

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