

# Beyond Polya'S Random Walk Theorem

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**Abstract.** A random walk can be regarded as a probability model depicting some degree of randomness, which has lots of interdisciplinary applications in physics, biochemistry and computer science. In this paper, the recurrence classifications of five different random walk models are presented along with their relevant studies. In order to explore the essential reasons leading to the qualitative change of simple random walks' recurrence property, the classification results of the five simple random walk variants are horizontally discussed. As a result, a positive bias is found to be the denominator shared by random walk variants whose recurrence classifications are different from that of simple random walks. A limited walking direction is also found useful in reversing the recurrence result. Besides answering the qualitative change question, this paper is also dedicated to provide a summary of the recurrence properties of different current random walk models, in order to help researchers in related fields to quickly get the picture of some random walk models.

**Keywords:** Random walk; recurrence; polya's random walk theorem.

## 1. Introduction

Consider a probably intoxicated person staggering and vacillating (unconsciously) among the directions, an empirically "random" path is created after successive steps, and the probability of finding this man at some distance from the starting point is required. This well-known random walk problem was firstly brought up by Pearson in 1905, which also brought the conception of random walk to the scene [1]. The further discussion of a long-term behavior of random walks was evoked by Polya. In his paper published in 1921, the random walk problem was set in the environment of  $R^d$  ( $d$ -dimensional Euclidean Space) with the restriction that the moving particle only came to rest at the integer lattice points, and above all, that the equal probability of taking each of the  $2d$  different orthogonal directions at each resting point. Such confined random walk got its name of Polya walk (simple random walk). Motivated by a series of embarrassing encounters, Polya carefully looked into the problem of two random-walker's collision, which reduced to requiring of the probability that a single random-walker return to its origin [2]. Polya's result, known as the Recurrence Theorem (Polya's Random Walk Theorem), reveals that with probability 1, a particle adopting 1-D or 2-D simple random walk will finally return to its starting point [3]. There is positive probability of chance, however, the particle will lose its way home in any higher dimensions [3].

After the work of Polya, increasing attention as well as importance was attached to random walk and its recurrence analysis. In the late 20 century, the rapid development of probability theory and stochastic process made a standard mathematical description of random walk possible. In the language of probability theory, suppose  $X_1, X_2, \dots$  is a countable series of i.i.d. random variables taking values in  $R^d$ , let  $S_n = X_1 + X_2 + \dots + X_n$ , then a (discrete) random walk is just the random variable sequence  $\{S_n; n \in N\}$ . With the tools in other branches of mathematics, a variety of proving methods to Polya's theorem were proposed, adding combinatorial [4], special function [5], stochastic method [6] and more perceptions to this classical theorem. Besides, new theoretical random walk models and their recurrence properties are investigated, generalizing Polya's theorem to random walks in other study-worthy environments with different restrictions.

The latest review that mainly focuses on the recurrence of random walk is done by Chen and Zhang in 2014, which only covers some results of simple random walk before that time [2]. However, numerous works have also been done over recent years in generalizing the properties of simple random walk to asymmetric [7], non-backtracking [8] and other more complex random walks. For

this reason, this paper is dedicated to providing a review covering the recurrence analysis of different random walk models as comprehensive as possible. Through this way, this paper tries to answer the historical question of what makes simple random walk recurrent in dimension 1,2 and transient in dimension 3. The remaining sections are arranged as follows. In section 2, notations and more details of simple random walk is provided along with the analysis of Polya’s random walk theorem. In section 3, the recurrence property of 5 types of simple random walk variants is presented, each with some discussion of its recurrence classification and relevant studies. In section 4, a summary of the recurrence property is included, along with the main contributions and limitations of this paper. Several open questions are also presented for further study in the conclusion section.

## 2. Basic Concepts of Simple Random Walk

### 2.1. Simple Random Walk

A standard description using the language of probability is presented in this section. For simplicity, in this paper, two states  $\epsilon_i = (\epsilon_i^1, \epsilon_i^2, \dots, \epsilon_i^d)$ ,  $\epsilon_j = (\epsilon_j^1, \epsilon_j^2, \dots, \epsilon_j^d)$  in  $Z^d$  are defined to be adjacent if  $\|\epsilon_i - \epsilon_j\|_1 = \sum_{t=1}^d |\epsilon_i^t - \epsilon_j^t| = 1$ , denoted as  $\epsilon_i \sim \epsilon_j$ , otherwise  $\epsilon_i \not\sim \epsilon_j$ . A  $d$ -dimensional simple random walk is a series of random variables  $\{X_n, n = 1, 2, \dots\}$  defined on the sample space  $S = Z^d$  with the one-step state transition probabilities.

$$p_{ij} = P(X_{n+1} = \epsilon_j | X_n = \epsilon_i) = \begin{cases} \frac{1}{2d} & \text{if } \epsilon_i \sim \epsilon_j \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

where  $\epsilon_i, \epsilon_j$  are two points in the sample space. Each random variable  $X_n$  can be interpreted as the location of the moving particle after  $n$  steps (or at the time  $n$ ). The 1, 2 and 3 dimensional simple random walks are illustrated by Fig. 1.

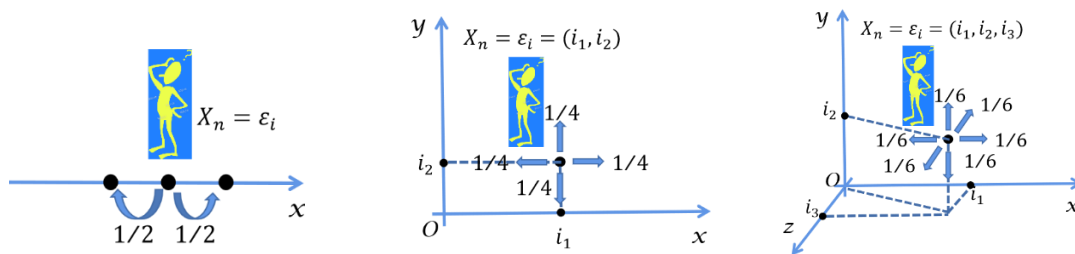


Fig. 1 Simple random walks in dimension 1, 2, 3.

Similarly, in standardized probability language, the concept of recurrence is described by the sum of some probabilities. Precisely, given random walk  $\{X_n, n = 1, 2, \dots\}$  (not necessarily be simple random walk), define  $f_{ii}(n)$  to be the probability of the moving starting from state  $\epsilon_i$ , returning to the same state for the first time with exactly  $n$  steps. Then consider.

$$F = \sum_{n=1}^{\infty} f_{ii}(n) \quad (2)$$

Since the sum of  $f_{ii}(n)$  also refers to the probability of finally returning to state  $\epsilon_i$ . Naturally, if  $F = 1$ , the state  $\epsilon_i$  is considered recurrent, otherwise ( $F < 1$ ) transient. If all states of the random walk are recurrent (transient), then this random walk is said to be recurrent (transient). If with positive probability, the moving particle can reach state  $\epsilon_j$  from state  $\epsilon_i$  in finite steps (or in time), then the state  $\epsilon_j$  is said to be accessible from state  $\epsilon_i$ . If two states are mutually accessible, then they commute. If two states  $\epsilon_i$  and  $\epsilon_j$  commute, then  $\epsilon_j$  is recurrent if and only if  $\epsilon_i$  is recurrent. With the lemma above, the problem of recurrence of a random walk can be transformed into recurrence of certain states. For random walks whose all states commute, for example, the simple random walk, the problem just reduces to investigate the recurrence and transience of an arbitrary state.

### 2.2. Polya’s Random Walk Theorem

Let  $u_{ii}(n)$  be the probability of returning to the original state  $\epsilon_i$  with exactly  $n$  steps (not necessarily for the first time). One defines.

$$U = \sum_{n=1}^{\infty} u_{ii}(n) \tag{3}$$

Polya proved that  $F = 1$  is equivalence to the divergence of  $U$ , which provides another criterion for recurrence. Hence the problem of computing  $f_{ii}(n)$  is transformed into a simpler calculation of  $u_{ii}(n)$ , which mainly consists of estimation of the series’ upper bound. Along with some integration techniques, Polya proved the following result. Polya’s Random Walk Theorem: Simple random walk is recurrent in dimensions 1 and 2, but transient in dimensions  $d \geq 3$  [3]. Actually, with the conditions of simple random walk, the expression of  $u_{ii}(2n)$  can be easily derived as

$$u_{ii}(2n) = \sum_{\substack{n_1, n_2, \dots, n_d \geq 0 \\ n_1 + n_2 + \dots + n_d = n}} \frac{(2n)!}{(n_1! n_2! \dots n_d!)^2} \left(\frac{1}{2d}\right)^{2n} \tag{4}$$

While  $u_{ii}(2n + 1)$  is always 0. There is a pleasing coincidence that the value of  $u_{ii}(2n)$  in dimension 1 just equals to the square of its 2D counterpart, indicating the simple random walk on the plane, to some extent, can be “broken down” into the walk on the lines. Unfortunately, this conjecture fails to hold for any higher dimensions ( $d \geq 3$ ). While new proof for Polya’s theorem is being constantly added [4-8], there is still one problem remains unclear today, i.e., what makes simple random walk jump from recurrence in the plane to transience in the stereoscopic space. This question also plays the role of meta-question in this paper.

### 3. Simple Random Walk Variants

For the remaining question, a natural conjecture is that the essential change occurs at some point during the transition from plane to three-dimensional space [9], which leads to two different ways of thinking about the question. The first one is to extend the simple random walk to continuous dimensions and carefully analyze the fractional dimensions between 2 and 3. The other method focuses on the restrictions of the random walk model. In answering the meta-question, this paper adopts the second way of thinking. Precisely, with the only focus on discrete random walk models, some changes of the recurrence conditions are observed while different restrictions are imposed on. And, if not mentioned otherwise, the sample spaces of the following random walks are all  $Z^d$ .

#### 3.1. State-Independent Random Walks

In this section, the recurrence of two slightly modified Polya walks are discussed.

##### 3.1.1 Symmetric Random Walks

Consider a random walk in  $Z^d$  (the sample space is  $Z^d$ ), then any two states  $\epsilon_i, \epsilon_j$  can be regarded as two lattice points in  $d$ -dimensional Euclidean space. Hence, two states are considered to be adjacent if and only if their corresponding vector  $\overrightarrow{\epsilon_i \epsilon_j}$  takes its value in  $\{\pm e_t, t = 1, 2, \dots, d\}$  where  $e_t$  forms the unit orthogonal basis of  $R^d$  (i.e.  $\epsilon_i \sim \epsilon_j$ ). A symmetric random walk is a random walk  $\{X_n, n = 1, 2, \dots\}$  with following state-transition probability.

$$p_{ij} = \begin{cases} \alpha_t & \text{if } \overrightarrow{\epsilon_i \epsilon_j} \in \{\pm e_t\}, t = 1, 2, \dots, d \\ 0 & \text{otherwise} \end{cases} \quad \text{ZEqn1} \tag{5}$$

where  $2 \sum_{t=1}^d \alpha_t = 1$ . Intuitively, for  $d$ -dimensional symmetric walk, the particle can only move to the adjacent state with the probability  $\alpha_t, t = 1, 2, \dots, d$ . The symmetry only remains for pairs of parallel vectors  $\pm e_t$ . Clearly, the simple random walk is just a special case of the symmetric random walk family with equal  $\alpha_t$ . The following result is given by Abramov in 2017 [7]. Recurrence of

Symmetric Random Walk. The Symmetric random walk is recurrent in dimension 1 and 2, and transient in dimensions  $d \geq 3$ . Abramov's work illustrates that neither "complete symmetry" nor "partial symmetry" is a sufficient condition in making simple random walks recurrent or transient. Actually, to reverse the recurrence classification, studies on 1-dimensional simple random walk variants such as Michelitsch et al [9]. and Ren's work [10] all indicates that a nonzero bias is necessary.

### 3.1.2 Random Walk with Conditional Rest

It is very natural to consider the likelihood of overstay when extending the symmetric random walk to a more general one. Following are two examples of this case. The family of random walk with 1<sup>st</sup> conditional rest is a kind of common random walk  $\{X_n, n = 1, 2, \dots\}$  with state-transition probability such that for any present state  $\epsilon_n$ , if  $\epsilon_n^t \neq 0$  ( $t \in \{1, 2, \dots, d\}$ ), then with probability  $\alpha_t$  the transition to the adjacent state  $\epsilon_n + e_t$  where  $2 \sum_{t=1}^d \alpha_t = 1$ . Otherwise, if  $\epsilon_n^t = 0$  ( $t \in \{1, 2, \dots, d\}$ ), then the probability of the adjacent state  $\epsilon_n + e_t$  being chosen equals to  $\alpha_t - \frac{\delta_t}{2} > 0$  where  $\delta_t > 0$ , satisfying the following condition.

$$0 < \frac{2\alpha_t - \delta_t}{2\alpha_t} \leq 1. \tag{6} \text{ ZEqn6}$$

Finally, with the complementary probability, the particle will stay at the present state. The family of random walk with 2<sup>nd</sup> conditional rest is a kind of common random walk  $\{X_n, n = 1, 2, \dots\}$  with state-transition probability such that for any present state  $\epsilon_n$ , if  $\epsilon_n^t = 0$  ( $t \in \{1, 2, \dots, d\}$ ), then with probability  $\alpha_t$  the transition to the adjacent state  $\epsilon_n + e_t$  where  $2 \sum_{t=1}^d \alpha_t = 1$ . Otherwise, if  $\epsilon_n^t \neq 0$  ( $t \in \{1, 2, \dots, d\}$ ), then the probability of the adjacent state  $\epsilon_n + e_t$  being chosen equals to  $\alpha_t - \frac{\delta_t}{2} > 0$  where  $\delta_t > 0$ , satisfying the following condition.

$$1 \leq \frac{2\alpha_t}{2\alpha_t - \delta_t} < \infty. \tag{7} \text{ ZEqn6}$$

From the definition above, one can easily see that a random walk with conditional rest is totally determined by two state-independent parameters  $\alpha = (\alpha_1, \dots, \alpha_d)$  and  $\delta = (\delta_1, \dots, \delta_d)$ . When  $\delta = \mathbf{0}$ , the random walk model with conditional rest just reduces to the symmetric one. The condition for recurrence is presented as follows. Recurrence of Random Walk with Conditional Rest. Random Walk with 1<sup>st</sup> and 2<sup>nd</sup> Conditional Rest is recurrent for  $d \leq 2$  and transient for  $d \geq 3$  [7].

### 3.2. State-Dependent Random Walks

In further extension, the determining parameters  $\alpha$  and  $\delta$  are considered variables changing along with history states. Considering a random walk  $\{X_n, n = 1, 2, \dots\}$ , it is called state-dependent random walk if at state  $\epsilon_n$ , the particle has probability  $L_t(\epsilon_{n-1}^t) \geq c > 0$  to occupy state  $\epsilon_n + e_t$  and moves to the adjacent state  $\epsilon_n - e_t$  with probability  $R_t(\epsilon_t) \geq c > 0$ , where  $c$  is a given constant ( $c < \min\{\alpha_i, i = 1, 2, \dots, d\}$ ,  $2 \sum_{t=1}^d \alpha_t = 1$ ) and  $L_t, R_t$  satisfy the following inequality.

$$L_t(\epsilon_{n-1}^t) + R_t(\epsilon_{n-1}^t) \leq 2\alpha_t \tag{8} \text{ ZEqn7}$$

With complementary probability  $1 - \sum_{t=1}^d [L_t(\epsilon_{n-1}^t) + R_t(\epsilon_{n-1}^t) \leq 2\alpha_t]$ , the particle will stay at its current point. The recurrence condition is given by Abramov. Recurrence of State-dependent Random Walks. State-dependent random walk is recurrent for  $d \leq 2$  and transient for  $d \geq 3$  [7]. Actually, the recurrence condition of Variant 3 implies the same result for as a generalization of above two cases since this variant is the extension of the two former random walk models. Moreover, the unchanged recurrence condition of the three variants also indicates that asymmetry, positive resting probabilities and state-dependence are not sufficient conditions for changing the recurrence classification of simple random walk.

### 3.3. Direction-Avoided Random Walks

Herdade and Vu proposed a new kind of simple random walk variant in which the particle keeps changing its moving directions, the recurrence classification of this model is carefully studied which yields surprisingly different results from the former variants. A direction-avoided random walk is a random walk  $\{X_{n,V}, n = 1, 2, \dots\}$  determined by a group of effectively  $d$ -dimensional unit vectors  $V = \{v_1, v_2, \dots, v_n\}$ , where “effectively” means that any hyperplane in  $V$  contains less than  $\lambda n$  ( $\lambda < 1$ ) vectors in  $V$  and  $v_i \neq v_j$  for any  $i \neq j$ . The proposition of the particle after  $n$  steps can be depicted as follows.

$$X_{n,V} = \sum_{t=1}^n \eta_t v_t \quad \text{ZEqn8} \quad (9)$$

Recurrence of Direction-Avoided Random Walk. Direction-avoided random walks are transient for all dimensions [11]. The transience for  $d = 1$  and  $d \geq 3$  are trivial, above all is transience in dimension 2. Besides proving the transience, Herdade and Vu also gives an accurate estimate of the upper bound of the returning probability for  $d = 2$ . Their result turns to be super polynomial, indicating that choosing different vectors for  $V$  is of no use to reverse the transience [11]. For  $d = 3$ , the upper bound is.

$$P(X_{n,V} = 0) \leq n^{-4+o(1)} \quad \text{ZEqn9} \quad (10)$$

Which also converges faster than simple random walk.

### 3.4. Non-backtracking Random Walks

A non-backtracking random walk is, intuitively, a random walk that the moving particle is forbidden to return to its last position. Study of such random walk model can date back to Alon et al., where the properties of non-backtracking walks are mainly studied in the perspective of graph theory [12]. Fitzner and Hofstad investigates the non-backtracking walk on infinite  $d$ -dimensional grid (i.e.,  $Z^d$ ) [13], and Kempton provides a perfect solution to the recurrence classification problem [8]. A non-backtracking random walk (on  $Z^d$ ) is a random walk  $\{X_n, n = 1, 2, \dots\}$  with following state-transition probability.

$$P(X_1 = \epsilon_1 | X_2 = \epsilon_2) = \begin{cases} \frac{1}{2d} & \text{if } \epsilon_1 \sim \epsilon_2 \\ 0 & \text{otherwise} \end{cases} \quad \text{ZEqn10} \quad (11)$$

$$p(X_{n+1} = \epsilon_{n+1} | X_n = \epsilon_n, X_{n-1} = \epsilon_{n-1}) = \begin{cases} \frac{1}{2d-1} & \text{if } \epsilon_{n+1} \neq \epsilon_{n-1} \text{ and } \epsilon_{n+1} \sim \epsilon_n \\ 0 & \text{otherwise} \end{cases} \quad (n \geq 2) \quad (12)$$

Recurrence of Non-Backtracking Random Walk. The non-backtracking random walk is transient for  $d = 1$  and  $d \geq 3$ , but recurrent for  $d = 2$  [8]. Jung and Markowsky further proved the equivalence of the recurrence of simple random walk and non-backtracking random walk on a regular infinite graph  $G$  of degree  $k \geq 3$  [14].

## 4. Limitations and Prospects

Table 1 illustrates the classification results covered in previous sections.

Back to the meta-question, the horizontal analysis adopted in this paper is far from discovering the essential reason accounting for the recurrence classification, but the work done in this paper is at least useful in confirming other relevant study by providing sufficient current models for further analysis. Actually, a more reasonable explanation about the recurrence classification problem is given by Beck [15] in his study of inhomogeneous random walk. Beck mentions in his paper that, the recurrence in a plane and the transience in the space are probability due to the value of expected distance after  $n$  steps  $\sqrt{n}$ , which means that almost all lattice points are visited in the  $\sqrt{n}$ -neighbor of the starting point on a plane while few are visited in the space. Since recurrence is equivalent to visiting each

state with infinite times. This explanation really provides a logical understanding of the jump from recurrence to transience. Besides the recurrence problem, studies on different random walk models and their other properties are always active, especially for non-backtracking model. A lot of work has been done in recent years in exploring deeper variants and properties of a non-backtracking-based PageRank [16, 17] as well as the models' combination with spectral theory [18]. Moreover, random walks models in random environments are paid increasing attention due to the model's higher level of randomness and authenticity in real-world process simulation.

**Table 1.** Results of Recurrence Classifications

Model	Recurrent dimensions	Transient dimensions
Simple Random Walks	$d < 2$	$d > 3$
State-independent Random Walks	$d < 2$	$d > 3$
State-dependent Random Walks	$d \leq 2$	$d > 3$
Direction-avoided Random Walks	$\emptyset$	$d > 2$
Non-backtracking Random Walks	$d = 2$	$d = 1, d \geq 3$

## 5. Conclusion

In this paper, the recurrence classification of mainly 5 types of random walk models is presented in trying to explain the qualitative change of simple random walk from dimension 2 to dimension 3. As the result, a biased (or inconsistent) state-transition probability is found to be necessary in reversing the recurrence classification of simple random walks. Some random walk variants with limited directions like Variant 4 and Variant 5 also show different recurrence properties, implying that attaching proper restrictions to directions of random walk model is also an effective way to change its recurrence classification. In more reasonable explanations, the expected-distance theory also provides another way to understand the meta-problem. Though the analysis method adopted in this paper is relatively basic, but this paper still acts as a brief review of current random walk models, being hopeful of providing useful research clues in the relevant field. All in all, the theoretical research is to form the basis of possible future applications. For the random walk model, its applications distribute in various fields across physics, biochemistry and computer science, demonstrating the inherent interdisciplinary nature of the random walk theory.

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