Set Theory and Third Mathematical Crisis: A Series of Events Reflected Upon the Mathematical Foundation

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Abstract. The basis of mathematics remains a highly debated and prominent subject among the mathematical community. The absence of a solid mathematical basis gives rise to a range of occurrences, such as mathematical crises. There is a multitude of arguments pertaining to the Mathematical Foundation. This research aims to elucidate the need for a foundation in mathematics and examine the origins of mathematical crises via an analysis of Classical Mathematics. Subsequently, an examination of three significant mathematical crises and the subsequent evolution of Mathematics over the 19th and 20th centuries will be undertaken. The Third Mathematical Crisis is a significant component of this work. Subsequently, a comprehensive exposition will be presented on the three prominent educational institutions that emerged during the Third Mathematical Crisis. This paper aims to provide a comprehensive analysis of the Third Mathematical Crisis, with a particular focus on the axiomatization of Set Theory. The process and outcome of this crisis will be thoroughly examined and discussed. There are counterexamples that challenge the notion of Set Theory as a basic theory. This paper will mostly concentrate on the subject of Category Theory. This paper provides an introduction to the genesis and identity of Category Theory, along with a concise explanation why the widespread interest among mathematicians in this field and its potential as a foundational framework for Mathematics. As a consequence, Category may serve as a fundamental basis to a significant degree.

Keywords: Classical Mathematics; Set Theory; Mathematical Foundation.

1. Introduction

In the modern educational system, mathematics seems to be one of the most important subjects. That's because many subjects are studied based on mathematical methods. There are also various cases that a new subject like Game Theory and Chaos Theory be created based on Mathematics. It also provides a new probability in some classic subjects. For instance, Newton's Mathematical Principles of Natural Philosophy had a significant influence on Physics hundreds of years ago, Philosophers now more likely to use symbolizations, which are also be created based on Mathematics, to describe logic. So, Mathematics could be called the foundation of many subjects. That means Mathematics plays a significant role in history. But there's a big problem that seems to be ignored and this problem is what's the foundation of Mathematics. Just like other subjects, Mathematics should have its own foundation to ensure its status, especially its various subjects' foundations.

In the mathematical area, the foundational question is also significant. Throughout history, there's always someone who likes to research it, especially at the end of the 19th century. After solving problems caused by irrational numbers and calculus, mathematicians noticed a new problem which is about the consistency of non-Euclidean geometry when it came up [1]. For this problem, the answer from mathematicians is that the consistency of non-Euclidean geometry relies on the consistency of Euclidean Geometry [1]. Mathematicians began with this geometric problem and noticed a more significant problem from the consistency problem. And this problem is about Mathematical foundations. In1870s, Set Theory was created by Cantor. He inputted various new identities for sets like cardinal numbers and enumerables. His thought made a significant impact on mathematics and mathematicians have seen Set Theory as a foundational theory at that time.

As a foundational theory, Set Theory are used to prove many theorems. Also, at the end of the 19th century, there was a major idea that Mathematics actually is a combination of loads of structures, and every structure is built on its own axiomatic system from a logical perspective [1]. Harold claimed that mathematical ideas are somehow arranged in strata [2]. For his thought, this is a kind of form that Mathematics may like to be. But that's only the response from a logical view. In addition, with its use, some paradoxes appeared. The appearance of these paradoxes caused the Third Mathematical Crisis to a great extent. Lots of mathematicians took part in it and aimed to solve this problem entirely. As a result, Set Theory is regarded as the foundation of mathematics again.

After this crisis, the Set Theory that is used to be the foundation is Axiomatic Set Theory. But there are still drawbacks in it and not all the paradoxes have been solved. During that crisis, only some hot paradoxes like Russell Paradox and Cantor Paradox. Some classic strata of Mathematics also can't be explained by Axiomatic Set Theory. So, in fact, this crisis still not end. There is also a trend that Category Theory may be more suitable than Set Theory to be the foundation in modern opinion. To study this topic, methods like analysis, deduction, and historical presentation will be used. This paper aims to find out more details about the relationships between Mathematics and its foundation, Mathematical Crisis, Set Theory, and Category Theory, and make a summary and conjecture about Mathematical Foundation.

2. The Necessity of Mathematical Foundation and Appearance of Set Theory

Throughout history, Mathematics has been a subject that was created from humans' need to count. That means Mathematics differs from Physics or Chemistry and it is abstract. So, foundations seem to be more important since Mathematics is entirely virtual. Thus, people have tried to find its foundation since Hellenic times.

2.1. Classical Mathematics and Origin of Mathematical Crises

Studying Classical Mathematics is valuable. For Classical Mathematics, it shows the beginning of mathematical development and various possibilities of Mathematics [3]. The Origin of Classical Mathematics mainly are Ancient Greece and China. For the Chinese situation, the greatest contribution in Mathematics is the perfect embodiment of Mathematical Application, especially in Arithmetic areas [3]. At that time, China needed more methods to deal with various realistic problems. For example, Nine Chapters on Mathematical Art only contains realistic problems, especially in agriculture, and provides mathematical methods to solve them. But there are few theoretical, or philosophical problems.

However, as Ancient Greeks had the foundation of philosophy to some extent. As there will always be some scary adjectives like irrational transcendent to be used in Mathematics. To solve that, the Ancient Greeks gave a typical opinion. The method is building Mathematics on a stable foundation to minimize this scare [4]. Also influenced by Plato, Aristotle, and Euclid these three mathematicians, Ancient Greeks thought the core of Mathematics was deduction. In the Hellenic view, Mathematics consists of logical deduction and proof. In the modern view, those Greeks made a big process of deductive systematization.

For the Mathematical Foundation, there is also a Hellenic explanation. At that time, the Pythagorean School thought that everything consisted of whole numbers. They see Arithmetic as the Foundation of Mathematics [4]. But soon, Hippias finds an irrational number, which is actually $\sqrt{2}$ and that causes the First Mathematical Crisis. But Euclid eventually created Euclidean Geometry by using five axioms to build this geometric system. $\sqrt{2}$ was explained as geometric quantity and Ancient Greeks thought that Mathematics was built on Geometry.

Unfortunately, there's a big gap after Hellenic times for the development of Mathematics. But after this gap, there's a big process for Mathematics. Leibniz and Newton invented Calculus and input the identities of limit and infinity and analysis replaces the status of geometry. But that may be a little fast for Mathematics cause everything we studied before this age is finite and it's visible. There weren't enough logical foundations to prove it is true like we hardly define the real notion of infinitesimal [5]. And that caused the Second Mathematical Crisis. With Mathematical Analysis becoming mature, this crisis was solved.

Apparently, these two crises all are caused by the lack of certain parts in Mathematics. After filling in these blanks, somehow people arrived at a new level and Algebra became the foundation again since mathematicians thought analysis was a kind of Algebra. However, it's still not stable enough to be the foundation. After these results, Cantor invented a Set Theory to fill the lack of foundation.

2.2. Appearance of Set Theory

Actually, there are rarely celebrities in academics like Cantor, not only because of his comments but also because he annoyed some mathematicians by disproving their thoughts. In the 18th century, people were more likely to see Mathematics as a tool to solve realistic problems. However, Cantor comes from a musical family, and compared with other mathematicians, he would like to pay more attention to romantic areas in Mathematics [5]. And that makes him successful.

In fact, the key to breaking down the problem in Second Mathematical Crisis is the Weierstrass function. And he is the key person that lets analysis become one of the most serious ways of mathematical deduction. Cantor was influenced by Weierstrass, Cantor aimed to study Pure Mathematics at that time. He posted his passages on Mathematics Annalen and Journal für Mathematik to show his work. In Cantor's view, a set is a collection of something that is identified and different [1]. For Contor, if an injection can be found between the part of the set and the entire set, then it will be infinite. With his ideas, he claimed that rational numbers set, and integer set are enumerable and real numbers set isn't enumerable.

For Set Theory, some great mathematicians held different ideas. Kronecker is one of the mathematicians who is against Cantor's idea. His idea is quite Hellenic, he thought that Arithmetic and Mathematical Analysis must be built on the whole numbers. His thought just holds the opposite opinion of Set Theory. Russell described Cantor's work as one of the greatest works of that age [1]. Hilbert said that Set Theory is the most amazing output of Mathematical Thought and one of the most beautiful behaviors in human activities for pure reasons.

Set Theory also explored many new areas at that time. Like Fractal Theorem, Topology Measure Theory and Group Theory are all created by applying Set Theory. Moreover, Topology is becoming more and more important in geometric areas nowadays, and the same as Group Theory in algebraic areas. End in here, Set Theory might be stable enough to be the foundation. It's absolutely good news for all mathematicians since their aims seem to have been achieved.

It should be clear that paradox comes from the limitation of people's thoughts and overthrowing it is the process by which people get rid of thoughts' limitations [6]. There's no surprise that Set Theory has its own paradoxes. These paradoxes deny it's foundational status to some extent and the foundation is lost again. Furthermore, these paradoxes caused the most serious mathematical crisis throughout history. Set Theory brings how many contributions human beings, and then its paradoxes trigger how many problems.

3. A Serious Mathematical Crisis

With the potential problems of Set Theory, the 20th century began. And the beginning of this century isn't so optimistic cause the potential problems turned into actual problems which are always called paradoxes. Then, the Third Mathematical Crisis broke out. This crisis, it's the most serious and complicated crisis compared with the other two and people would like to divide this crisis into three major parts. From the find of paradoxes to the appearance of three major schools in Mathematical Philosophy. And Axiomatic Movement, which is the most important part of this crisis, is among the entire crises.

3.1. Paradoxes of Set Theory

At that time, Burali-Forti Paradox, Cantor Paradox emerged in succession and those three paradoxes are regarded as the most significant paradoxes. In the Burali-Forti Paradox, it is said that all the sequences of ordinal numbers are well-ordered, the ordinal number it has should be the maximum of all ordinal numbers [1]. It means that ordinal numbers will form a well-ordered set A based on natural order and according to identity, set A would have an ordinal number B and B is a member of A. But according to the ordinal number's identity, B should be bigger than any ordinal number of A, so it shouldn't be A's member [7]. That's the contradiction of one perspective about Set Theory, and it also attracted Cantor's attention at that time.

However, Cantor has noticed this problem in 1895. After 4 years, he asked Dedekind can all cardinal numbers themselves formed a set to point out that the set that contains all sets can't be talked in letters [1]. Let Y be the set that contains all sets, and then the powerset of Y should be Y's subset. That means the cardinal number of P(Y) isn't greater than the cardinal number of Y. But according to Cantor's Theorem which he found in 1891, for any set, the cardinal number of its powerset must be greater than its cardinal number. Obviously, there's a contradiction between his new find and his old theorem. To solve this, he tried to use a method about consistent sets and incompatible sets to deny it. But the Burali-Forti Paradox denies the thought of denying the Cantor Paradox.

In parts of Cantor's thought, there's something similar to Russel's paradox. When Cantor gave the definition of a set, he only explained that a set is a kind of collection. But there's no limitation about which should, and which shouldn't be a member of the set. And this problem becomes the Russel Paradox. Maybe the Barber Paradox is more familiar than the Russel Paradox, but they are the same paradox essentially. The argument is that there is a set A which contains all elements except itself and the question is can A be the member of A? If A is the member of A, then A doesn't include A. Otherwise, if A isn't the member of A, then A should include A, then A is the member of A. Apparently, whether from this view, there always be a contradiction that A should include itself. In fact, the Russel Paradox is seen as the most important cause it finds a problem in basic mathematical logic.

3.2. Three Major Schools

Almost every mathematician has a unique opinion on those paradoxes, but we still can sort them into three major schools. The first and probably the most important one is logistics school. One of the most important problems about Mathematical foundations that can't be handled is the relationship between Mathematics and logic [8]. For logistic school, its core argument is Mathematics is an extension of logic rather than a single subject. Frege first had this thought in 1884, and Russel and Whitehead first created a logistic system and created Mathematics in this system in Principia Mathematica. They also claimed two main points. One is mathematical identity can be proven from logistic identity by accurate definition. The second one is mathematical theorem can be proven from logistic axiom by pure logistic deduction. Their main aim is to find a serious logistic structure for Mathematics.

The second popular school at that time was the Intuitionist School. Kronecker, one of the biggest challengers of Cantor, created this school and Brouwer spread it. For intuitionists, implanting mathematical identity comes before languages, logic, and experience [1]. In their ideas, Mathematics is the product of people's spirits [9]. Mathematical theorem and identity should be justified and defined by structures seriously. They even don't want to admit irrational numbers. According to Brouwer, he thought that everything including analysis should base on integers, and that a quite like the Pythagorean School. Another main point is that logic relies on Mathematics rather than Mathematics is the extension of logic. In their opinion, logic is a kind of language, and it just provides a rule to combine words and words to talk about axioms, but it can't be axioms [1].

However, Hilbert used to be seen as an intuitionist, but this was soon denied. Intuitionists thought there would always be something unknown because of the limitation of our thoughts, but Hilbert thought there is no ignoramus in Mathematics [1]. Finally, he created the Formalist School. In his opinion, if people want to input some imaginary elements like infinity or limit, they should be proofed in logic. Symbolization and formalization seem to be good methods. It is a bit like Logistic School's idea, but it should be noticed that the most visible difference is that Logistic School wants to use logic to define Mathematics. But for a Formalist School, must find a way of formalization to justify Mathematics.

3.3. Axiomatic Movement

It isn't hard to feel that whether which school, they all want to find a system for Set Theory. Then back to the view of Set Theory, it can be said that the lack of a system caused the appearance of paradoxes in Set Theory. At that time, Axiomatization became a trend. One of the reasons why geometry is so stable might be they have their own axiomatic systems. Either Euclidean or Non-Euclidean Geometry, they both have their own axioms to describe the change in their areas. This is the embodiment of axiomatization and undeniably, these axioms make an axiomatic system for shape changes.

As axiomatization makes the strictness of Geometry, mathematicians have the thought that it is also possible to build an axiomatic system for Set Theory as it is a part of Mathematics. Peano, an Italian mathematician, used to give an axiomatic system. In his opinion, we can only use the identity of 0 and successor to gain any numbers in Arithmetic. But some philosophers like Russel or Frege, would say those are just the truth of logic. It should also be clear that the original problem set can be a member of itself.

In 1908, Zermelo, a German mathematician first claimed the thought of axiomatic Set Theory. To get rid of the contradiction of the set which contains itself, he put forward 7 axioms to be the necessary limitation for Set Theory and made a system called the Zermelo System [8]. In 1922, Frankel found the difference between the attribute of the set and the set itself. He used this find to supply and improve Zermelo's theory and finally, ZF system (Zermelo-Frankel System) was created. There's something that should be noticed that there isn't AC in the ZF system because it is disputed, but AC is the basis of lots of scientific theorems, especially for subjects like Abstract Algebra [4]. So, if there's AC in the ZF system, it is called the ZFC system, which is the final result of the axiomatization of Zermelo and Frankel.

The axiomatization of Set Theory seems to solve these known paradoxes successfully, all mathematical identities can be explained by the language of Set Theory in the ZF system. In addition, von Neumann, Bernays, and Gödel also created an axiomatic system which is the NBG system for Set Theory, but every theorem in ZF can also be the theorem in NBG. The formalization of Axiomatic Set Theory creates a new area for Mathematics.

4. Perspectives on Set Theory and Category Theory

Though axiomatization improves the strictness of Set Theory, and it also seems to have solved all known paradoxes about Set Theory, it still has problems. There are already quite a lot of categories of mathematics in which sets can't defined even after axiomatization. Most counter opinions are from formalists, and they give a new theory which is Category Theory to replace the foundational status of Set Theory.

4.1. Limitation of Set Theory

Paul Benacerraf, a formalist, says that numbers only express an abstract structure and there's no set that can express a number, but in the language of Set Theory, $\{\varnothing, \{\varnothing\}\}$ or $\{\{\varnothing\}\}$ are used to define number 2 [10]. Many mathematicians think that singletons are meaningless. Singleton is a kind of set that only contains one member. It hardly be seen as a collection cause Singleton only has one member, but it may seem to be useful. For example, if there's no singleton, the result of $\{a,b\}\cap\{b,c\}$ may be undefined if there's no identity of unit sets [11]. But here's only for intersection, which is also a part of Set Theory.

One of the main applications of Set Theory is mapping. Mapping is a mathematical way which is used to define functions. For a function, there are three important elements, domain, co-domain, and function, and function is the relationship between domain and co-domain. Suppose that domain and co-domain are both singletons and the members of each of them are specific, the mapping which we want to study is also specific. That means it’s unnecessary to study mapping in that case.

To talk about mapping, there’s also a big shortage in Analytic Geometry for Set Theory. After Descartes created the coordinate system, people tended to put geometric shapes like lines and curves in this system and describe them by equations. It is quite useful since the attribute of a geometric shape may not explained perfectly in natural language. In this analyzing method, we can use the equation $x+y=1$ to describe a line, and everything about this line like the slope or the mapping between domain and codomain will be expressed directly. However, curves will be more complex. Some of the curves may be simple like $f(x)=2x$ or $f(x)=\ln x$, we still can find a mapping between its domain and co-domain. But if a circle should be expressed by the equation, it is x2+y2=R2 (R stands for the radius of the circle), which means for one specific value of x, there will be two specific values of y. That makes a contradiction with the identity of bijection and that also means it can’t be explained by Set Theory.

For another view, the Set Theory also seems to be useless in the area of applied mathematics. Even though in this area, there’s a lack of theoretical proof, it’s still an important part of Mathematics. In this range, people mainly use mathematical methods like statistics and Possibility Theory to study worlds like economic problems or problems in natural scientific areas rather than study abstract items. Also, the trend of change will always be studied in this area. It is also important to do some comparisons like comparing various countries' GDP and doing some analysis in Applied Mathematics. So, tables or graphs may be much more useful than sets. The set is just a collection, it can't do anything to describe the changes or show some comparison directly.

4.2. A New Hope: Category Theory

Although mathematicians use axiomatization to create an Axiomatic Set Theory to solve the paradoxes, but it’s still imperfect. And this means that Set Theory will never be the foundation of Mathematics. For a structure, the best case is that there’s only one instance to describe it. However, if a structure has various instances, the attributes of the structure may not be described clearly [12]. in 1945, a new theory, Category Theory, was claimed by Samuel Eilenberg and Saunders Mac Lane.

As mathematicians want to explain Mathematics in structure, undeniably set is a strong structure, but the only element in the set and their relationship will be mainly studied. Category is a special case of structure [12]. One of the biggest differences between Set Theory and Category Theory is that Set Theory pay more attention to mathematical objects, instead, Category Theory pays more attention to the relationship between mathematical objects [13]. Categories consist of the classes of sets and some functions. Sets and functions are both special examples of categories. Also, in Grothendieck’s view, there are small categories and big categories in Category Theory [13]. For small categories, they can be expressed by sets, but big categories can’t. This shows the transcendence of Category Theory over Set Theory.

Another advantage of Category, Theory is that it’s Hilbertian. In Category Theory, some jargon like morphisms, functors, and objects don’t have specific meanings, they only can be explained in certain axiomatic systems. That means categories themselves don’t rely on circumstances, and they can be used as structures. Also, just like sets can be related by functions, the same structures can be related by the functions that are stored in the structure in Category Theory, and these functions are called morphisms. Functions that stay in the attribute of category can relate category and category together, and these functions are called functors. Also, there may be a thought that morphisms and functors can form a category that involves all categories. This problem is probably like the problem ‘the sets which involve themselves'. The best way to solve it is still axiomatization. But it doesn’t mean there will be an Axiomatic Category Theory, main purpose is to let the identities of categories themselves be axiomatic.

In 1966, Ravel claimed a kind of axiomatization, and when Ravel's Axiomatization is limited to the dispersed category, the axiomatization of Set Theory is in category form. 3 years later, he came up with the Topos Theorem when he described a Type Theory about categories. And there was a new way to explain sets that set form a topos, so it’s too simple to discuss the complexity of subjects like Topology or Algebraic Geometry. Also, in this theorem, a logic can be found, and the final results of this logic equals the logic of the Intuitionist School. Category Theory began with the formalization of a structure, and its result can provide logic in intuition, and during the whole process, logic is everywhere. That can prove that Category Theory may be the solution which all three schools will be accepted.

5. Conclusions

Foundation has been an important part of Mathematics from ancient to the present. From the birth of Mathematics, the foundation seems to have already been set. But it’s still unknown. From the integers to building a strict structure for mathematical deduction and finding a perfect axiomatization, the events about the foundation are uncountable. Throughout history, there are being three crises in Mathematics that are caused by the lack of mathematical foundation. Mathematicians will always be keen on solving these problems and various mathematicians become famous after claiming some new ideas about that. And totally these serious can be divided into two parts and the border is the 19th century. Before the 19th century, people were first influenced by the Pythagorean School, they thought everything in our lives consisted of whole numbers. Soon irrational numbers destroyed the possibility of this thought. After the gap of the Middle Ages, the appearance of Calculus led the development of Mathematics to a new stage. However, the rationality of Calculus caused a new problem. After solving this problem, Analysis became popular, and it also brought a new possibility for Mathematics. After the 19th century, Cantor claimed Set Theory. As a theory that provides a new structure, people’s thoughts about mathematical foundations become deep to some extent. Mathematicians find the importance of structures for Mathematics. Unavoidable, paradoxes appeared and there is a more serious crisis broke out. Logistic School, Intuitionist School, and Formalist School were founded separately in this crisis. Finally, almost all mathematicians chose to use axiomatization and Zermelo-Frankel Set Theory became one of the most famous and useful Axiomatic Theory. But it still doesn’t solve the paradoxes from fundamental. At the latest time of mathematical development, Category Theory appeared. The appearance of Category Theory becomes a surprise for mathematicians and in lots of strata, Category Theory always can play an important role. There are always lots of schools that think Category Theory is the foundation of Mathematics. But it still doesn’t become the universe and there still isn’t enough evidence to prove it can be the foundation. But the possibility of Category Theory is quite positive from most mathematicians’ view, and it may be proofed perfectly and accepted one day.

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