The Debates on Infinity: A Mathematical History Approach

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Abstract. Calculus and set theory sparked centuries of debate on infinity, which continues today. After discovering paradoxes and inconsistencies, mathematicians and philosophers questioned the underlying systems and conceptions of the infinite. Today, we easily use the infinity sign in our academic work. We often forget infinity's turbulent debut in our contemporary use of the term. David Hilbert wrote "On the Infinite" in the 1920s to persuade sceptics to embrace and use infinity. Gödel and his second incompleteness theorem defeated him very quickly. Beyond Gödel's claim of system inconsistency, Hilbert's theory neglected two factors when seen from a current viewpoint. First is the genuine nature of actual infinity, and second is Zermelo-Fraenkel's (ZF) axiom-based refined set theory. This article will look into the these great achievements and argue the deficiencies of them.

Keywords: Gödel incompleteness theorem; Zermelo–Fraenkel set theory; Infinity.

1. Introduction

With the advent of Calculus and set theory, the notion of infinity became a subject of enduring controversy and discourse spanning centuries, persisting into the contemporary era. The concept of infinity faced severe criticism upon the revelation of paradoxes and contradictions, prompting mathematicians and philosophers to challenge the existing foundational systems and theories [1-3]. Nonetheless, in the present day, we readily embrace the concept of infinity and effortlessly incorporate the infinity symbol into our academic work.

In the current utilization of infinity, it tends to overlook its tumultuous reception upon its initial introduction. In the 1920s, David Hilbert authored the article "On the Infinite" with the intention of substantiating the merits of embracing the concept of infinity to those skeptical opponents, aiming for broader acceptance and usage [4]. However, his efforts were promptly rebuffed by Gödel and his second incompleteness theorem [5]. Nevertheless, beyond Gödel's assertion of the inconsistency of systems, there remain two aspects in Hilbert's work that were inadequately addressed or partially explored when reconsidered from a modern perspective [6, 7]. The first pertains to the true nature of actual infinity, while the second involves the refined set theory based on Zermelo-Fraenkel's (ZF) axioms [8].

David Hilbert wrote the book "On the Infinite" in the 1920s with the intention of convincing sceptics to accept and make use of infinity. His work was published under the title "On the Infinite." This was an undertaking that he completed successfully. Godel and his second incompleteness theorem proved to be too much for him to handle, and he was swiftly and easily vanquished. The triumph was achieved in a very rapid amount of time. In addition to Godel's claim that the system is inconsistent, Hilbert's theory ignored two key considerations, which may be perceived as problematic when seen from a contemporary perspective. The first is the true character of real infinity, and the second is the axiom-based refined set theory that Zermelo and Frankel (ZF) developed. Both of these concepts are related to Zermelo and Frankel's work. Both of these ideas are connected to the notion of infinity. The concept of "stage theory," which is a framework for systematically dividing sets into different stages for the goal of enhancing clarity, was thought of and improved by Boolos. People might perhaps accept the notion of "stage theory" for further growth. This theory is a framework for methodically separating sets into various phases. Within the boundaries of this framework, he proposed that the foundational stage ought to be referred to as stage zero and ought to only be constituted of the empty set. This was his working hypothesis. On the other hand, following stages come into existence as a consequence of an iterative or inductive amalgamation of groups of people (who may be numbered indefinitely) and the entirety of sets that were previously formed at antecedent stages. This process may be thought of as a recursive or inductive process. This methodical order starts with the very first step, which is numbered zero. This is the beginning of the development. Because of the evidence that has been presented in this article, it seems that the arguments over infinity have reached a ceasefire for the purpose of the further development of set theory and its descendants. However, this state of affairs will only be transitory when more mathematical progress is made.

2. The definition of infinity and its puzzle

To commence, people must delve into the definition of infinity. Philosophers have grappled with this inquiry since the 19th century, spawning numerous rigorous discussions aimed at classifying generalized infinity into two sub-categories. Hilbert introduced the distinction between actual infinity and potential infinity in his work "On the Infinite," a demarcation that led to deeper contemplations on the matter of consistency. He defined actual infinity as "a totality of simultaneous existence," implying that it embodies complete possibilities constituting an interval [4]. Conversely, potential infinity relates to the incomplete, often surfacing in the realms of analysis and calculus. Concepts of infinite largeness and smallness remain somewhat elusive, accessible in approximation but elusive in absolute attainment. These divergent categories entail distinct implications and approaches to the same question. For instance, from a potential perspective, if we were to divide an object with a definite volume into "infinite" pieces, we would conclude that this division is possible, with each piece occupying a volume infinitesimally close to zero, albeit not precisely zero. This notion aligns with the infinitesimal concept fundamental to calculus. On the contrary, actual infinity disputes this conceptual construction endorsed by the potential approach. Hilbert concentrated on empirical observations and physical laws, contending that no tangible entity can be infinitely divided. His studies encompassed liquids, electricity, and energy, culminating in the assertion that all real-world objects or substances inherently possess a minimum unit: metals and liquids possess atoms, whereas electricity and energy involve electrons and quanta [4]. The unequivocal declaration is that the realm of actual infinity, as introduced by Cantor, exclusively resides in the numerical domain, as delineated by Hilbert – an extension of natural numbers or the real number interval between zero and one [4]. Consequently, actual infinity becomes a conceptual construct, a tool to address gaps and intricacies in nascent theories and systems. Thus, the prudent utilization of actual infinity and set theory necessitates additional substantiation and evidence of consistency.

However, Hilbert’s formulation of actual infinity was not without its imperfections. His approach to consistency was constructed upon the conceptual nature of infinity, yet there exist counterexamples that challenge the real-world implications of actual infinity. In Levey’s article within The Philosophical Review, Leibniz suggested that actual infinity could be conceptually achieved through the division of a moving object during its interval of motion [9]. In modern times, contemporary concepts such as spacetime points have further affirmed Leibniz’s concept. The foundational underpinning of this notion involves the establishment of two distinct relations. The first encompasses the relationship between distance and time, while the second involves the occupancy relation. To elucidate the occupancy relation, the creation of two distinct types of particulars becomes essential, with one type capable of bearing the occupancy relation in relation to the other, and this yields infinite possible combinations between the two relations (SEP). The example of spacetime points distinctly contradicts Hilbert’s assertion that actual infinity is confined solely to the domain of numbers, even though the concepts of space and time lack tangible attributes in reality. Moreover, it serves to undermine the argument that actual infinity is inherently counter-intuitive, given the seamless and natural construction of spacetime and its associated occupancy relation.

Moreover, Hilbert's proposition, focusing exclusively on establishing the consistency within the domain of finite numbers, encountered rejection from Gödel. This rejection was accompanied by Gödel's second incompleteness theorem, a demonstration that no system could self-prove its own consistency. Consequently, the question pertaining to the harmonization of infinity with a finite scope and the reliability of set theory persists, marked by a landscape riddled with multiple paradoxes and contradictions. Foremost among these challenges is Russell's paradox, a celebrated conundrum that starkly interrogates consistency, particularly when viewed independently from Gödel's second incompleteness theorem. This paradox engenders a substantial breach within the fabric of the theory and the very notion of actual infinity it encompasses. The crux of Russell's paradox can be succinctly encapsulated in the following statements:

There exists a set containing all sets that do not include themselves.

There is no set containing all sets that do not include themselves.

Both of these statements are upheld by the axioms of Cantor's set theory, paradoxically validating its inconsistency even within a rudimentary formulation.

This had a direct impact on the concept of actual infinity, initially introduced through Cantor's exploration of set countability and bijections, which in turn prompted skepticism regarding the reliability of this novel notion within the framework of an inconsistent theory. However, in more recent times, Ernst Zermelo and Abraham Frankel devised a new axiomatic system aimed at eradicating the core issues posed by Russell's paradox. In Boolos' exposition titled "The Iterative Concept of Set," the foundational concepts of Zermelo-Fraenkel (ZF) set theory were delineated, shedding light on why these axioms are considered intuitive and inherently natural by contemporary scholars. Boolos explained, that rather than" t blocks their derivation by artificial technical restrictions on the set of axioms that are imposed only because paradox would otherwise ensue", ZF's axiom focuses more on the new conceptual construction of the set, which "is not only a consistent (apparently) but also an independently motivated theory of set" [10].

Boolos developed the concept of "stage theory," a framework aimed at systematically classifying sets into distinct stages to enhance clarity. Within this framework, he postulated that the foundational stage is designated as stage zero, exclusively comprising the empty set. Subsequent stages, on the other hand, emerge via the iterative or inductive amalgamation of collections of individuals (numbered infinitely) and the entirety of sets previously formed at antecedent stages. This sequential progression originates from the initial stage zero [10]. Employing this approach, it becomes readily apparent that no set can include itself. This insight stems from the observation that the constituents of each set predominantly consist of individuals or sets generated during earlier stages. Essential axioms of ZF such as separation, extensionality, pairs, and infinity find their conceptual underpinnings within the stage theory framework. A lack of contradictions is ensured by this intuitive method, as it inherently and coherently delineates the hierarchical arrangement of sets, thereby dispelling any perplexities that may arise. Thus, it was not required that the contradictions of set theory necessitated an alternative approach for establishing consistency.

3. Gödel's theorem and its reflection upon incompleteness and consistency

It has now attained a robust comprehension of the fundamental framework underpinning ZF set theory, coupled with a well-established mastery of the intuitive principles that underscore the stage theory. Nevertheless, there persists a substantial void that warrants our meticulous consideration: Gödel's first and second incompleteness theorems.

To embark upon this discourse, it is imperative to delineate two pivotal definitions. First and foremost, it must elucidate the concept of completeness, and secondly, it must expound upon the notion of consistency. Consistency, in this context, signifies that within a coherent system, we can indeed establish the proof of either a statement or its negation. Essentially, if we can demonstrate the veracity of a statement, it implies that, under the condition of system consistency, we are precluded from simultaneously establishing the veracity of its negation. Completeness, conversely, signifies that within a comprehensive system, we possess the capability to establish the proof of any statement endowed with a determinable truth value.

Gödel's first incompleteness theorem presents a rigorous demonstration that no consistent formal system can achieve completeness. This implies that within any such system, where all statements neither yield contradictions nor can be definitively confirmed or refuted, there invariably exist statements that possess truth value but elude formal proof. Gödel's proof of this theorem was ingeniously founded on the concept of Gödel numbering, which he employed to encode all mathematical statements into numerical representations, thus facilitating the integration of logical and axiomatic statements. To encode logical symbols, he employed prime numbers, ensuring that each mathematical statement corresponded uniquely to a natural number, rendering them easily decodable. Initiating with a solitary statement and its associated Gödel number, Gödel established a mechanism for substituting this statement into others possessing distinct Gödel numbers but equivalent meanings. This process culminated in the formulation of a self-referential statement, denoted as "G," which implies its own unprovability.

Paradoxically, however, the formal system can employ other statements bearing different Gödel numbers to demonstrate the provability of "G." This paradox hinges on a perplexing conundrum: If "G" is provable, then it must be false, as it asserts its own unprovability. Conversely, if "G" is unprovable, then it must be true. This seminal proof conclusively establishes the existence of at least one statement, "G," which simultaneously holds truth and defies no formal proof within a given system. Furthermore, the addition of new axioms proves ineffective in resolving this paradox at its core, as it merely sets the stage for the emergence of analogous statements, perpetuating the cycle of undecidability.

Gödel's first incompleteness theorem posits the existence of true statements that elude deduction from the presently established set of axioms. Intriguingly, these axioms themselves may also be nondeductible. Consequently, conjectures arise within the context of incomplete formal systems, suggesting that they fail to encompass all independent nondeductible statements as axioms. A notable illustration can be found within the realm of Euclidean geometry, where the system relies upon a set of postulates. The omission of any one of these postulates invariably leads to manifest incompleteness, characterized by correct statements devoid of formal proof. In essence, statements rooted in axioms or other statements must inherently be deductible, amenable to proof through deductive steps.

Thus, it logically follows that existing theorems and postulates, even those seemingly conceived from pure abstraction, such as the concept of actual infinity derived from ZF set theory, can potentially be regarded as true without giving rise to paradoxes or contradictions [11]. This holds true as long as the foundational constructs of the stage theory remain robust. Notably, most of ZF's axioms, with the exception of the axiom of extensionality, can be deduced from the theorems rooted in stage theory. Leveraging iterative principles and the notion of successors, the concept of stages, along with the holistic notion of infinity and the assignment of an ordinal number to each stage, emerges organically and intuitively. Importantly, this framework remains devoid of contradictions and is impervious to Gödel's first incompleteness theorem [12].

To delve further into the intricacies of Gödel’s Second Incompleteness Theorem, it is imperative that it should first elucidate its underlying essence and outline the sketch of its proof. It is germane to revisit Gödel’s First Incompleteness Theorem as a foundational context. In that theorem, we unequivocally established the notion that within coherent and expressive formal systems, inherent incompleteness persists. It is of paramount importance to underscore that our entire logical edifice, including proofs and inferences, is rooted in the foundational premise of the system's inherent consistency.

In the prior exploration, this stud succeeded in ascertaining the capacity to formulate what is famously known as a Gödel statement, or, equivalently, a Gödel number—a construct imbued with the remarkable attribute of self-reference. This construct asserts its own truth while remaining impervious to formal proof. It is noteworthy that the veracity of such a statement serves as an implicit indicator of the system's continuity and coherence, albeit bereft of a formal demonstration. This profound revelation engenders a profound consequence: a formal system can lay claim to the assurance of its own consistency only if it paradoxically harbors inconsistency. This paradox is evocative of our earlier illustration, wherein a statement represented by the symbol G simultaneously yields proof and refutation. In summation, the proof of Gödel’s Second Incompleteness Theorem is inextricably derived from the foundational proofs and principles laid down by the First Incompleteness Theorem.

4. Conclusion

Calculus and set theory were the sparks that ignited a debate over infinity that has persisted for millennia and is still going strong to this day. Mathematicians and philosophers started rethinking the underlying systems and concepts of the infinite after the apparent discovery of inconsistencies and paradoxes. The use of the infinity sign in the academic work that individuals undertake nowadays is so commonplace that it's almost automatic. The stormy origin of infinity is sometimes overlooked in the way that we use the concept in current times. David Hilbert wrote the book "On the Infinite" in the 1920s with the goal of persuading sceptics to accept and make use of infinity. He was successful in this endeavour. He was quickly and easily defeated by Godel and his second incompleteness theorem. The victory came in a very short length of time. When seen from a modern point of view, Hilbert's theory neglected two significant concerns, in addition to Godel's argument that the system is inconsistent. The first is the genuine nature of real infinity, and the second is the axiom-based refined set theory that Zermelo and Frankel (ZF) devised. Both of these concepts pertain to infinity. For more development, people could possibly embrace the idea of "stage theory" which is a framework for methodically dividing sets into various phases for the purpose of improving clarity, was conceived and refined by Boolos. Within the confines of this structure, he hypothesised that the foundational stage should be referred to as stage zero and should just be composed of the empty set. On the other hand, subsequent stages come into existence as a result of an iterative or inductive amalgamation of groupings of persons (which may be numbered endlessly) and the whole of sets that were previously produced at antecedent stages. This logical progression begins at the very first step, which is numbered zero. As these has been illustrated in this article, the debates over infinity seem to have a truce for the current development of set theory and its successors. However, it only remains temporary as the mathematics marches forward.

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