On the Continuity of Functions and Their Application in Polynomial Approximation

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Abstract. Researchers continue to explore the topological properties of continuity, including the use of open sets, neighborhoods, and limit points to define and characterize continuous functions. For other aspects, scholars have made achievements in both linear operators and functionals in the context of functional analysis, including applications to integral and differential equations. The development would also see achievements in studies with partial derivatives, directional derivatives, and differentiability. This study primarily discusses how to understand the continuity of one-variable and multivariable functions and introduces some methods for determining continuity. It then explores the application of function continuity in polynomial approximation, specifically focusing on approximating continuous functions in different scenarios using polynomials.

Keywords: Continuity; Polynomial; Functional Analysis.

1. Introduction

The continuity of functions enjoys a very important position in mathematical analysis because continuous functions have very good properties [1-3]. However, the logic of the human brain is often different from the strict mathematical definition, which may lead to some deviations or misunderstandings in the cognitive process of these concepts. For example, in the initial contact with the concept of "continuity", people usually use colloquial forms such as "it rains all day long" to indicate that the rainfall is uninterrupted and continuous [1].

This has led some people to understand the continuity of a function as complete and without gaps. In 1981, a study of 41 university students examined their judgment of the continuity of a function image, and all of them chose to judge whether a function is continuous or not of notion intuitive: whether a function consists of a single expression, whether a function can be plotted or not, whether a function is complete or not, and so on, instead of judging it from a strict definition [1].

It is also interesting to note that the concept of continuity of a function was also not articulated in a survey of 15 teachers with undergraduate degrees in mathematics, one of whom, giving the concept of functional continuity at the certain point, wrote that "as n gets closer to n_0, $f(n)$ gets closer to $f(n_0)$" [2]. Such a description is obviously one-sided and imprecise. This study will discuss the different definitions of continuity and consistent continuity of unitary and multivariate functions, as well as their applications in the field of mathematics.

Given a neighborhood $U(n_0)$ of $f(n)$ at n_0, then define $f(n)$ to be continuous at n_0 if and only if $\lim_{n\to x0} f(n) = f(n_0)$. From this clear and strict mathematical definition, we return to the previous statement about the continuity of the function being one-sided. While it is true that "as n gets closer to n_0, $f(n)$ gets closer to $f(n_0)$" implies the meaning of the above expression, it is not entirely true. An example can be given here: $f(n) = \begin{cases} n^2, n \neq 0 \\ -0.001, n = 0 \end{cases}$. In this example, as n keeps approaching 0, $f(n)$ is indeed approaching $f(n_0)$, but $\lim_{n\to 0} f(n) = 0 \neq -0.001$, it is not continuous at 0 although it fulfills the conditions of that statement. So mathematical formulae often have deeper mathematical meanings, and it is not possible to summarise the exact meaning of a formula in one or two sentences.

2. Definition of continuity and complex cases derived

By such a strict definition of continuity, people can use the limit in the $\varepsilon-\delta$ language to rewrite this definition, one gets.

$$\forall\varepsilon>0,\exists\delta>0,|n-n_0|<\delta,s.t.|f(n)-f(n_0)|<\varepsilon$$

This definition is the most common one, but of course there are two variants under this definition:

$$\forall V\big(f(n_0)\big),\exists U(n_0),s.t.f(U(n_0))\subset V(f(n_0))$$

$$\forall\varepsilon>0,\exists U(n_0),\forall n\in U(n_0),|f(n)-f(n_0)|<\varepsilon$$

Where $U(n_0)$ represents the neighbourhood of n_0 and $V\big(f(n_0)\big)$ represents the neighbourhood of $f(n_0)$ [10]. Both definitions are actually rewrites of the initial definition, using more abstract concepts to facilitate subsequent mathematical learning. After clarifying the definition of continuity, people can use it to determine whether a function is continuous or not. The next two examples to further familiarise we with its process: 1. $f(n)=\sin n$ is continuous on R.

Proof: $\forall n_0\in R\ \forall\varepsilon>0,\exists\delta>0,|n-n_0|<\delta,s.t.|f(n)-f(n_0)|<\varepsilon$

This is the general form of the proof, and it is only necessary to find one δ that matches the above formula, then the proof is complete.

According to the analysis, it would have the condition $|f(n)-f(n_0)|<\varepsilon$, so the basic idea is to introduce the relationship between n and x0 from this inequality.

$$|\sin x-\sin x0|=|2\cos\frac{n_0+n}{2}\ \sin\frac{n-n_0}{2}|\leq\ |2\sin\frac{n-n_0}{2}|2\sin\leq\ |n-x0|<\varepsilon$$

From this, it follows that δ is sufficient to take a number not greater than ε.

2. This is a very famous function called the Riemann function:

$$R(n)=\begin{cases}\frac{1}{n},n=\frac{m}{n}(m,n\in N^*,m,n\text{ are coprime numbers})\\ \qquad 0,n\in R\backslash Q\cup\{0,1\}\end{cases}$$

This function looks slightly complicated, but $R(n)$ is continuous at any irrational point on $(0,1)$, as will be shown next.

That is, it is necessary to prove that.

$$\forall\varepsilon>0,\exists\delta>0,s.t.|n-n_0|<\delta,|R(n)-0|<\varepsilon$$

The results of the approximation of n by irrational and rational points are discussed below in two cases:

In case an irrational n exists, then $|R(n)-0|=0<\varepsilon$;

In case a rational n exists, $|R(n)-0|=\frac{1}{n}$.

Next step is to prove that $\frac{1}{n}<\varepsilon$.

It is clear from the observation that there are only a finite number of points that satisfy $\frac{1}{n}\geq\varepsilon$, denoting all the points that satisfy as $n_1,n_2,\ldots,n_N$.

Take $\delta=\min\{|n_1-n_0|,|n_2-n_0|,\ldots,|n_N-n_0|,|0-n_0|,|1-n_0|\}$, this step digs out all the points that satisfy $\frac{1}{n}\geq\varepsilon$ and control δ to be satisfied in the interval $(0,1)$, then it can be obtained as $\frac{1}{n}<\varepsilon$.

In summary, it follows that n, whether approximated by a rational or irrational point, satisfies $|R(n)-0|<\varepsilon$, so $R(n)$ is continuous at any irrational point.

After discussing continuous functions, I would like to introduce the approximation of functions by polynomials in this context. Here is a brief introduction to a very famous formula: the Taylor formula, if one wants to approximate $f(n)$ at a point by a polynomial, one only needs to satisfy the nth-order derivability of $f(n)$ to obtain the approximating polynomial.

$$f(n) = \sum_{t=0}^{n} \frac{f^{(t)}(z)}{t!}$$

This case requires a function to be derivable, and in unitary functions, derivability is a stronger condition than continuity, so if a function is not derivable but continuous, can it be introduced that this function can be approximated by a polynomial? Weierstrass approximation theorem gives the answer. Theorem: If $f(n)$ is a continuous function on the closed interval $[z, c]$, then for any $\varepsilon > 0$, there exists a polynomial $P(n)$ such that $|P(n) - f(n)| < \varepsilon$.

With this theorem, it shows that no matter how complex a polynomial is, it can be simplified by a simple polynomial. In addition, it does not require the property that the function is derivable, which ensures that it will have a wider range of applications.

The Weierstrass Approximation Theorem states the fact that continuous functions can be approximated by polynomials. Next, the study will introduce a polynomial to show exactly how it approximates continuous functions.

3. Chebyshev polynomials

N-algebraic polynomials of the form $\cos(n \arccos x)$.

First prove that $\cos(n \arccos x)$ is an nth-degree polynomial, using mathematical induction.

When $n = 1$, $\cos(\arccos x) = n$,holds.

When $n = 2$, $\cos(2x) = 2\cos^2 n - 1$ and $\cos(2 \arccos x) = 2n^2 - 1$, holds.

The assumption holds when $n = k, n = k - 1$, and when $n = k + 1$,

$\cos(kx + n) = \cos(kx)\cos x - \sin(kx)\sin x$

and by the assumption, $\cos(kx)$ and $\cos(kx - n)$ can be expressed as algebraic polynomials, so it is only necessary to show that $\sin(kx)\sin x$ can be expressed as an algebraic polynomial.

Since $\cos x$ and $\cos(nx)$ can be expressed by polynomials, $\sin(kx)$ can be expressed by polynomials too. Then the proof finished.

Before it is clear that such polynomials can be used to approximate continuous functions, it is necessary to understand the properties of such functions.

First of all, this function has a value range of $[-1,1]$, $|T_n(n)| = 1$ has exactly $n + 1$ solutions, here is a short proof:

$$\cos(n \arccos x) = 1 \rightarrow n = \cos\frac{2k\pi}{n}, k = 0,1,\ldots, n - 1$$

This yields the number of solutions when n is even $\frac{n+2}{2}$, the number of solutions when n is odd is $\frac{n+1}{2}$.

$\cos(n \arccos x) = -1 \rightarrow n = \cos\frac{2k\pi+\pi}{n}, k = 0,1,\ldots\ldots, n - 1$, it can be obtained when n is even the number of solutions is $\frac{n}{2}$ and the number of solutions when n is odd is $\frac{n+1}{2}$.

In summary, the number of solutions to $|T_n(n)| = 1$ is $n + 1$ whether n is odd or even.

Next prove the final conclusion that $\frac{T_n(n)}{2^{n-1}}$ is the polynomial with the smallest deviation from 0.

Proof: The converse method is used here by assuming that there exists a first 1 polynomial $P_n(n)$ such that $\exists n$, $|P_n(n)| < 2^{1-n}$

By the previously studied property, the $n + 1$ solutions of $|T_n(n)| = 1$ are denoted by size as $n_1, n_2, \ldots, n_{n+1}$.

Discuss the case when n is even: $P_n(n_1)<\frac{T_n(n_1)}{2^{n-1}}$, $P_n(n_2)<\frac{T_n(n_2)}{2^{n-1}}$, ... and so on gives $n+1$ similar inequalities (same when n is odd), leading to $F(n)=P_n(n)-\frac{T_n(n)}{2^{n-1}}$ has n zeros. However, $F(n)$ is an $n-1$ polynomial with at most $n-1$ zeros, so there exists a contradiction and the assumption is not valid.

By this theorem, it can have a good approximation to polynomials, an example of which is given below:

Find the four best approximations of $f(n)=2n^5+6n^4+2n^3+n^2+2x+5$ on $[-1,1]$.

The expression for $T_5(n)$ is given here first: $16n^5-20n^3+5x$, then $P_4(n)=f(n)-2^{-3}\,T_5(n)=6n^4-0.5n^3+n^2+1.375x+5$.

After discussing the continuity, the study will introduce another kind of continuity that has stronger conditions than the determination of continuity: uniform continuity. Let it first clarifies its definition [3]:

$$\forall\varepsilon>0,\exists\delta>0,\forall n_1,n_2\in I,|n_1-n_2|<\delta,s.t.|f(n_1)-f(n_2)|<\varepsilon$$

(Here, 'I', including those appearing below, is an interval) This has similarities to the concept of continuity, but a major difference is that continuity is defined for a single point, whereas uniform continuity is defined for the totality of points in an interval, which also means that δ depends on the value of the point and ε in the continuity, whereas the value of δ depends only on the value of ε in the uniform continuity. A simple example is given here: $f(n)=n$, which we prove to be uniformly continuous on R using the most primitive definition:

$$\forall\varepsilon>0,\exists\delta=\varepsilon,\forall n_1,n_2\in I,|n_1-n_2|<\delta,s.t.|f(n_1)-f(n_2)|=|n_1-n_2|<\varepsilon$$

Of course, because this function is more specific, so to use the most primitive definition of a good deal, for complex functions it will introduce more simple methods to determine.

A lemma is introduced before presenting the next theorem with Cauchy Sequence [3].

Lemma: A sequence $\{a_n\}$ and if $\forall\varepsilon>0,\exists N,s.t.n>N,m>N,|a_n-a_m|<\varepsilon$.

The Cauchy Sequence plays a significant role in mathematics, as it can be used to determine whether a sequence or infinite series is convergent or not. We will discuss its application to continuous functions, and the following is a corollary that follows from the Cauchy Sequence:

Corollary: $f(n)$ is attributed as uniformly continuous on I, only in one case in which given two sets$\{X_{n1}\}(n_{n1}\in I)$, $\{X_{n2}\}(n_{n2}\in I)$ and $\lim\limits_{n\to\infty}(n_{n1}-n_{n2})=0$, then $\lim\limits_{n\to\infty}f(n_{n1})-f(n_{n2})=0$.

This corollary gives a method used to prove or prove not of non-uniformly continuity, i.e.,To have two subsequences on the region of the function that satisfy $\lim\limits_{n\to\infty}(n_{n1}-n_{n2})=0$ but $\lim\limits_{n\to\infty}f(n_{n1})-f(n_{n2})$ is not 0, is workable. Two examples will be given below for better understanding:

1. $f(n)=\frac{1}{n}$ is non-uniformly continuous on $(0,1)$ because there exists a series $X'_n=\frac{1}{n}\left(n'_n\in(0,1)\right)$, $X''_n=\frac{1}{2n}\,(n''_n\in(0,1))$, satisfying $\lim\limits_{n\to\infty}\left(\frac{1}{n}-\frac{1}{2n}\right)=0$, but $\lim\limits_{n\to\infty}f(n_{n1})-f(n_{n2})=\lim\limits_{n\to\infty}(-n)\neq0$.

2. $f(n)=\sin\frac{1}{n}$ is non-uniformly continuous on $(0,1)$ for a series $X'_n=\frac{1}{2n\pi}(n'_n\in(0,1))$, $X''_n=\frac{1}{2n\pi+\frac{\pi}{2}}$ (xn’’∈(0,1)), satisfying $\lim\limits_{n\to\infty}(\frac{1}{2n\pi}-\frac{1}{2n\pi+\frac{\pi}{2}})=0$, but $\lim\limits_{n\to\infty}f(n_{n1})-f(n_{n2})=1$.

In addition to proving non-uniform continuity by using the construction of series, it is also possible to judge by using limits [4].

For $f(n)$ is in $[a,+\infty)$ is uniformly continuous ($a\in R$) and $\lim\limits_{n\to+\infty}\frac{g(n)}{f(n)}=0$, then $g(n)$ is uniformly continuity there.

For $f(n)$ is in $[a,+\infty)$ is non-uniformly continuous($a\in R$) and $\lim\limits_{n\to+\infty}\frac{g(n)}{f(n)}=+\infty$, then $g(n)$ is non-uniformly continuity there.

These two theorems are also very powerful for determining uniform continuity, we only need to know whether some elementary functions are uniformly continuous, and then we can get the answer whether some complex functions are uniformly continuous. For example, by the previous continuity proof of that $f(n) = n$ is on R, then $f(n)$ satisfies uniform continuity on $[a, +\infty)$ for any a. And we also have $\lim_{n\to+\infty} \frac{\sqrt{n}}{n} = 0$, so is y=$\sqrt{n}$ on R.

In addition to this, people can also use the derivability to determine whether a function is uniformly continuous or not, as given here firstly in the Lipschitz Theorem [5].

If $f(n)$ satisfies the Lipschitz condition on the interval I, i.e., there exists a constant L > 0 such that for any $n', n'' \in I$, there is $|f(n') - f(n'')| \leq L|n' - n''|$, then $f(n)$ is uniformly continuous on I.

It requires that $|n' - n''| = \frac{\varepsilon}{2L}$, and then we can make $|f(n_1) - f(n_2)| < \varepsilon$.

And if this theorem is combined with the Lagrange median theorem, the following equation is obtained:

$|f(n') - f(n'')| = |f'(\xi)|n' - n''| \leq L|n' - n''|$($\xi$ between n' and n'')

The corollary that $f(n)$ has uniformly continuity here if $f(n)$ is derivable from the same interval with bounded derivative function.

So, with the above theorem, people have transformed the problem of uniform continuity of a function into a problem of finding the derivative to determine the range of the derivative function.

And there is also a very famous theorem: the Cantor-Heine theorem [3].

Functions that are continuous on a closed interval are uniformly continuous on this interval.

It only needs to determine whether a function is continuous or not, and in the previous discussion, the continuity of a function is relatively easy to determine, together with the condition of closed intervals, to know the uniform continuity of a function.

In summary, by introducing tools such as sequences, limits, derivatives, and continuity to determine the uniform continuity of a function, we can see that the mathematical concepts are all interchangeable and it is easy to simplify the problem.

Similarly, it could introduce a mathematical application under consistent continuity, i.e., Bernstein polynomial approximation of a consistent continuous function [6].

For $f(n) \in C\ [0,1]$, $g_n(n)=\sum_{k=0}^{n} C_n^k n^k (1-n)^{n-k} f(\frac{k}{n})$

$$\forall\varepsilon > 0, \exists n > N, s.t. |g_n(n) - f(n)| < \varepsilon$$

Proof: Since f is uniformly continuous, it will have

$$\forall\varepsilon > 0, \exists\delta > 0, \forall n_1, n_2 \in I, |n_1 - n_2| < \delta, s.t. |f(n_1) - f(n_2)| < \frac{\varepsilon}{2}$$

$$g_n(n) - f(n) = \sum_{k=0}^{n} C_n^k n^k (1-n)^{n-k} f\left(\frac{k}{n}\right) - f(n)$$

$= \sum_{k=0}^{n} C_n^k n^k (1-n)^{n-k} (f\left(\frac{k}{n}\right) - f(n))$.

$|g_n(n) - f(n)| \leq \sum_{k=0}^{n} C_n^k n^k (1-n)^{n-k} |f\left(\frac{k}{n}\right) - f(n)|$

$=\sum_{|\frac{k}{n}-n|<\delta} C_n^k n^k (1-n)^{n-k} |f\left(\frac{k}{n}\right) - f(n)| + \sum_{|\frac{k}{n}-n|\geq\delta} C_n^k n^k (1-n)^{n-k} |f\left(\frac{k}{n}\right) - f(n)|$

=I1 +I2

For the first part, by using the property of uniformly continuous, it can get.

$I1 \leq \sum_{k=0}^{n} C_n^k n^k (1-n)^{n-k} \frac{\varepsilon}{2} = \frac{\varepsilon}{2}$

Since f has uniformly continuity and is defined on a closed interval, there will be T, such that $|f(n)|$<T.

$I2 \leq 2M\sum_{|\frac{k}{n}-n|\geq\delta} C_n^k n^k (1-n)^{n-k} \leq 2M\sum_{k=0}^{n} C_n^k n^k (1-n)^{n-k} (\frac{k}{n} - n)^2 \frac{1}{\delta^2}$

$=2M\sum_{k=0}^{n} C_n^k n^k (1-n)^{n-k} (\frac{k^2}{n^2} - 2\frac{k}{n} n + n^2) \frac{1}{\delta^2}$

Let it be separated into three parts.

$$\frac{2M}{\delta^2}\sum_{k=0}^{n} C_n^k n^k (1-n)^{n-k} \frac{k^2}{n^2} = \frac{2M}{\delta^2}\sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} n^k (1-n)^{n-k} \frac{k^2}{n^2}$$

$=\frac{2M}{\delta^2}\sum_{k=1}^{n-1} \frac{(n-1)!}{(k-1)!(n-k)!} \ (\frac{k-1}{n} + \frac{1}{n})\ n^k (1-n)^{n-k}$

$=\frac{2M}{\delta^2}\sum_{k=2}^{n-2} \frac{(n-2)!}{(K-2)!(n-k)!} \frac{n-1}{n} n^k (1-n)^{n-k} + \frac{2M}{\delta^2}\sum_{k=1}^{n-1} \frac{(n-1)!}{(k-1)!(n-k)!} \frac{1}{n}\ n^k (1-n)^{n-k}$

$=\frac{2M}{\delta^2} (\frac{n-1}{n} n^2 + \frac{1}{n} n)$

$(2) - \frac{2M}{\delta^2}\sum_{k=0}^{n} C_n^k n^k (1-n)^{n-k} 2\frac{k}{n} n = -\frac{4M}{\delta^2}\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} n^k (1-n)^{n-k} \frac{k}{n} n$

$=-\frac{4M}{\delta^2}\sum_{k=1}^{n-1} \frac{(n-1)!}{(k-1)!(n-k)!} n^{k-1}\ (1-n)^{n-k} n^2 \ = -\frac{4M}{\delta^2} n^2$

$(3)\ \frac{2M}{\delta^2}\sum_{k=0}^{n} C_n^k n^k (1-n)^{n-k} n^2 = \frac{2M}{\delta^2} n^2$

(1) +(2) +(3) = $\frac{2M}{\delta^2}\left(\frac{n-1}{n} n^2 + \frac{1}{n} n\right) - \frac{4M}{\delta^2} n^2 \ + \frac{2M}{\delta^2} n^2 \ = \frac{2M}{\delta^2} \frac{n(1-n)}{n}$

Since n(1-n)$\leq \frac{1}{4}$, we have (1) +(2) +(3)$\leq \frac{M}{2n\delta^2}$

If n is big enough, then we can assume that $\frac{M}{2n\delta^2} < \frac{\varepsilon}{2}$

Finally, it can be concluded that $|g_n(n) - f(n)|$<ε.

So, not only can continuous functions be approximated using polynomials, but functions that are uniformly continuous can also be approximated using their properties to construct polynomials. Finally, this article generalizes the continuity of functions from unitary to multivariate. However, the discussion of the continuity of functions under multivariate becomes more complicated because what we discussed before was done on R. Making the generalization leads to R^2 , R^3 and so on. So, it will introduce the continuity of multivariate functions by introducing the notion of metric spaces.

4. The continuity of multivariate functions

Definition: Let X be a set if for any two points a and b in X, the distance from a to b is denoted by $d(e,f)$ satisfying the following conditions: (1) $d(e,f) = 0$ if and only if $e = f$ [6].

(2) $d(e,f) = d(f,e)$

(3)$\forall$ c $\in$ X, $d(e,f) \leq \ d(e,g) + d(f,g)$

Then X is said to be a metric space.

It is easy to see that $R^n (n \in N^+)$ are all metric spaces. Therefore, we can use the concept of metric space to unify the definition of the continuity of multivariate functions, which is defined as follows [7].

Let A and A_1 be two metric spaces and $E \subseteq A$, x0 be a point in E.

$$\forall\varepsilon > 0, \exists\delta > 0, d_a(n, n_0) < \delta, s.t.\ d_Y\big(f(n), f(n_0)\big) < \varepsilon$$

Then continuity is proved.

Of course, using this definition in multivariate functions would look awkward, so I'll introduce some ways to determine the continuity of a multivariate function from other perspectives. Firstly, we can transport some properties of unitary functions to multivariate functions, for example, a function at n_0 is continuous if and only if the max or min value of it at n_0 is equal to the value of the function in one-variable functions, then similarly, the same property is true in multivariate functions.

Secondly, the Lipschitz condition we introduced earlier can also be utilized for multivariate functions, and here is an example for binary functions [8]. If $f(n,y)$ is continuous for every component and Lipschitz continuous with respect to a component (e.g., n).

Based on the previous discussion of the Lipschitz condition, we can understand that $f(n, y)$ is continuous in the domain of definition as long as it is continuous with respect to each component and there exists a component such that the derivative function derived from it is bounded in the domain of definition. Let us understand the above statement by giving an example: $f(n, y) = ex + \sin y$ at $\forall(n, y) \in R^2$ is continuous on R because $f(n) = e^n$ is continuous on R and $f(y) = \sin y$ is continuous on R and $|f'(y)| \leq 1$.

There is also another way of discriminating that is very similar to the one mentioned in consolidation [7]. Again, $f(n, y)$ is required to be continuous for each component, but $f(n, y)$ is judged to be continuous in the domain of definition as long as it is monotonic in the domain of definition for one of the components. This gives us an alternative view of the above example, namely that $f(n) = e^n$ is monotonically increasing on R, so $f(n, y)$ is continuous.

Then, for the three methods mentioned above, which are similar to one-variable functions, we think about the judgement of the continuity of multivariate functions from the point of view of limits and derivatives.

Similarly, consistent continuity under one-variable functions is generalized in multivariate functions [9]. Let f be a mapping from the metric space X to the metric space Y if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall n_1, n_2 \in X \text{ s.t. } d_X(n_1, n_2) < \delta, d_Y\big(f(n_1), f(n_2)\big) < \varepsilon$$

In one-variable functions, we introduced a very important theorem i.e. a function is uniformly continuous if it is continuous on closed intervals, and the extension to multivariate i.e. a function is uniformly continuous if it is a tight set on a metric space X and the function is continuous. Extended to n-dimensional Euclidean space, all bounded closed intervals can be understood as being tight sets. From this, it can be seen that there is also a very close connection between the uniform continuity of a one-variable function and the uniform continuity of a multivariate function.

In multivariate functions, the same polynomials can be used for approximation. We can generalize the Bernstein polynomials. Recall that its form in unitary functions $\sum_{k=0}^{n} C_n^k n^k (1-n)^{n-k} f(\frac{k}{n})$, a generalization to N elements gives [10].

In multivariate functions, it similarly requires that the domain of definition of each component be controlled between $[0,1]$, but it is not necessary for the function to be uniformly continuous, and the above approximation to $f(n_1, \ldots, n_N)$ can be obtained only if the function is continuous.

5. Conclusion

As a final observation, this article discusses the concepts of continuity and uniform continuity, highlighting their close connections with limits, derivatives, and other related concepts. It also explores their applications in polynomial approximation. Furthermore, it extends the discussion of continuity from univariate functions to multivariate functions by utilizing the methods and techniques employed in analyzing univariate functions, thereby establishing a connection between the discussions on continuity in both contexts.

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