

From Set Theory to the Axiomatization of Set Theory

Jingying Jiang *

Shenghua Zizhu Academy, Shanghai, China

* Corresponding Author Email: jane.jiang@stu.ahu.edu.cn

Abstract. Axiomatic set theory was created by German mathematician Zermelo as a strategy for addressing and resolving paradoxes in the field of mathematical study. By adopting this strategy, the axiomatic technique will be applied to set theory. The person argues that Cantor's failure to impose restrictions on the idea of a set is the cause of the dilemma. They also claim that Cantor's definition of a set is unclear. Both of these arguments are predicated on the idea that Cantor neglected to place limitations on the idea of a set. Zermelo hypothesized that the condensed version of the axioms would make it easier to define a set and elaborate on its properties, and he was correct in his prediction. The creation of an axiomatization for set theory is the first of this research's main goals. Second, the investigation of various set theory development methods in comparison to one another. Even though there are intriguing puzzles in the field of set theory that have not yet been solved, the axiomatization of set theory is widely regarded as a significant accomplishment in the field.

Keywords: Set theory; Axiomatization; Paradox.

1. Introduction

A large part of modern mathematics can be based on set theory using axiomatic systems, and everyone who studies and uses mathematics should have a basic understanding of set theory [1]. This series' goal is to give a fundamental overview of set theory using the axiom system. A fundamental area of mathematics is set theory, which examines general sets. Set theory holds a special place in mathematics, and its fundamental ideas have permeated all areas of the subject. The mathematical theory of sets, often known as set theory, includes the most fundamental mathematical notions of sets, elements, and membership [2, 3]. Set theory provides the vocabulary for describing mathematical objects used in the majority of contemporary mathematical formulations. The axiomatic foundation of mathematics is formed by set theory, logic, and first-order logic, which formally constructs mathematical objects in terms of ill-defined concepts like "set" and "set member" [4-6].

A set is viewed as a self-proving idea, such as a collection of items, in naive set theory. In axiomatic set theory, sets and set members are first defined by axioms that specify their attributes rather than being immediately defined. According to this theory, rather than being explicitly specified, sets and set members are represented as points and lines in Euclidean geometry. The creator of set theory and a notable German mathematician at the end of the 19th and beginning of the 20th centuries was Cantor. one of the most creative and divisive thinkers in mathematical history. His studies on continuity and infinity at the end of the 19th century substantially departed from the accepted use and meaning of infinity in mathematics, sparking intense arguments and even stern condemnation [7].

However, the development of mathematics finally proved Cantor right [8]. His set theory is regarded as the greatest mathematical creation of the 20th century. The idea of a set has significantly broadened the area of mathematical study and given structure to the subject. Set theory has had a significant impact on contemporary philosophy and logic in addition to modern mathematics. He established set theory in works published between 1872 and 1893. He first put forth the idea of an infinite set in these writings, demonstrated the uncountability of the set of real numbers, advanced the "superfine number" theory, and advanced the "continuum hypothesis" that there is no base in between the base of the set of natural numbers and the base of the set of real numbers. Cantor suggested the idea of set equivalence as a way to extend the idea of the number of items in a finite set to an infinite set. It is established that a straight line and the common N-dimensional space correlate one to one. "I found it, but I couldn't believe it," he said, marveling at the result. Cantor completely altered the organization of mathematics, thoroughly disclosed the fundamental characteristics of the

infinite, and encouraged the emergence and growth of new mathematical subfields. The "Russell Paradox" appeared in 1903, just as mathematicians were celebrating the creation of a solid mathematical basis. Russell's paradox had an impact on set theory, logic, and even the very core of mathematics. Hilbert declared that this paradox had disastrous consequences for mathematics [9]. Even though the non-contradiction of non-Euclidean geometry and the basis of calculus theory eventually became the study of set theory, if set theory emerged logical shield, then the reliability of geometry and calculus will be worried.

2. Difficulty Revisited

Russell Paradox divided all sets into two classes: (1) normal sets, characterized by the fact that the set itself cannot be treated as a single element. For example, a human society composed of one person is not one person; Not all collections of natural numbers are natural numbers. (2) An abnormal set, characterized by an element that is also of its own. The set of all conceptions is still a concept, just as the set of all sets is still a set. In addition, there are examples of abnormal sets in real life. For example, a collection of all markets is still one large, unified market; A consortium of all libraries can still be regarded as one library. Is S a normal set, an abnormal set, a subset of itself, or neither if S is a set of all normal sets? If $S \in S$, then S is an abnormal set, so $S \notin S$; If $S \notin S$, then S is a normal set, so whether $S \in S$ is a normal set or not leads to a contradiction. Therefore, S belongs if and only if S does not belong to S . After the crisis, many mathematicians devoted themselves to the work of solving the crisis. In 1908, German mathematician Zermelo put forward an axiomatic set theory, trying to eliminate paradoxes by the axiomatic method of set theory. According to him, the paradox exists because Cantor does not define set specifically and his definition is ambiguous [10].

Zermelo hoped that the simplicity of the axioms would make the definition of a set and its properties more obvious [11]. Since then, the intuitive concept of the set has been established based on strict axioms, thus avoiding the emergence of paradoxes. Axiomatic set theory is the second phase in the development of set theory. Naive set theory, on the other hand, refers to Cantor's set theory that existed before 1908. Naive set theory is strictly handled by axiomatic set theory. It preserves the valuable achievements of naive set theory and eliminates its possible paradoxes, thus solving the third mathematical crisis satisfactorily. The direct purpose of axiomatizing set theory is to eliminate paradoxes, that is, to hope that set theory established according to axiomatization is harmonious and does not produce logical contradictions.

The study of axiomatic set theory can help us to understand set theory more deeply, can help us build mathematical models to solve some complex mathematical problems, and can also help us to apply the knowledge of other mathematical fields more easily. In addition, axiomatic set theory can be applied to computer science, such as the design of programs and algorithms, system programming, and the design of computer systems. To exclude super-large sets such as "the set of all sets" and avoid contradictions, German mathematician Zermelo put some necessary restrictions on set cooperation. He put out these seven axioms: (1) Extension axioms: If $S = T$ and $T = S$ for sets S and T , then S equals T . (2) The empty set axiom states that there is a set that has no elements. There exists the set a , a , b for elements a and b in any set. (3) The separation axiom states that for a set S in which the function is definite $p(x)$, there exists a set T that only contains those $x \in S$ for which $p(x)$ is true. (4) The power set axiom states that $P(S)$, the power set of S , is a set if S is a set. (The set of all S 's subsets is the power set of S .) The union of S is a set if S is a set, according to the fifth axiom of union. Sixth selection axiom: There exists a subset T of the union of S that shares exactly one element with each member of S if S is a nonintersection of a nonempty set. (7) The axiom of infinity states that for every item a , if $a \in Z$, $a \in Z$. This set Z contains empty sets. Paradox avoidance in axiomatic set theory The self-reference, negation idea, and connection to totality and infinity in the definition of the set are the main sources of the paradox in set theory. According to Kring, a definition of a "non-predicate" is one in which the set M and a particular object m are defined by the technique, on the one hand, m is an element of M , and on the other hand, the definition of m depends on M . In contrast, the "direct

predicate definition" is that when a definition of a whole object is to be given, the predicate that defines the whole object must be completely independent of the whole content of the object to be defined. From a logical point of view, nonpredication is self-winding or circular, because what is defined has already permeated the definition.

Russell pointed out that no population can contain the following two elements: it can only be defined by the population, and it either includes or presupposes the population. But there are many non-predicate definitions in classical mathematics, the famous mathematician Weyl has made great efforts to clear or rewrite these definitions, although fruitful, but not completely solved.

3. The Construction of the Axiomatic System

Zermelo does not distinguish between sets and properties of sets, which are treated as having the same meaning [11]. The Zermelo axioms became the present standard Zermelo known as the Frank axiom system, also known as the ZF system, in 1922 after Frank discovered the distinction between the set's attributes and the set itself. This system adds the axiom of regularity, the axiom of existence of unordered pairs of sets, and the axiom of substitution. Since the contentious axiom of choice leads to the so-called "sphere-splitting paradox," which states that a sphere can be divided into two spheres with the same radius after a finite division, ZF systems typically do not incorporate it. This result contradicts human experience and classical mathematics. However, the axiom of choice is the premise of the basic theorems of many subjects, especially in abstract algebra. If the selection principle is added, it is called a ZFC system. Zermelo's axiomatic system of sets, which made up for the shortcomings of Cantor's naive set theory, successfully eliminated all known set theory paradoxes. In addition, several mathematicians have also proposed and established different axiomatic set theories. Von Neumann, for example, argued that paradoxes occur not because sets are too large, but because they are used as elements of other sets. Just don't admit that any set can be an individual and become an element of another set. According to von Neumann's analysis, paradoxes in set theory are not caused by recognizing the status of a class as a set, but by treating it as an element of another class. He distinguishes between "class" and "set", and a class is a set so large that it cannot be included in other sets or classes. A set is a class limited to elements that can be used as a class. In this case, a collection is a safe class. The theory developed by mathematicians such as von Neumann, Bernays, and Godel is known as the NBG system or BG system. Later mathematicians discovered that any theorem in the ZF system is also a theorem in the BG system, and vice versa.

The construction of axiomatic set theory has opened up a new platform for the foundation of mathematics. So, does axiomatic set theory avoid paradoxes altogether? Poincare said, "The sheep are fenced against wolves, but we do not know whether there is a Wolf in the enclosure." In his book *Introduction to Metamathematics*, the famous mathematical logician Klein also pointed out that "paradoxes are avoided in the axiomatization of hypothesis set theory - we can only be assured of this in the negative, that is, we have not yet encountered paradoxes." Although no contradiction has been found in the ZF system so far, the problem of its non-contradiction has not been conclusively proved, so it is still uncertain whether new paradoxes will appear. In a certain sense, axiomatic set theory simply bypasses paradoxes and does not "solve" them. Conclusion In the first half of the 20th century, when faced with the flaw in Russell's paradox in set theory, Russell, Brouwer, Hilbert, and other mathematical giants proposed various solutions, resulting in the formation of three renowned schools of mathematical philosophy: logicalism, formalism, and intuitionism. The Logician school believes that the whole meaning of mathematics can be deduced from pure logic, without relying on any particular mathematical concept. The formalist school believes that all mathematics can be obtained by operating on the rules expressed in mathematical forms and that the basic object of mathematical thinking is the symbols of mathematical forms themselves, not the mathematical meanings borrowed from them. Intuitionism does not agree with the unrestricted use of the law of exclusion and opposes the reductive method of proof. In 1922, Hilbert outlined a comprehensive plan to rebuild the mathematical foundation, to demonstrate the "completeness," "consistency," and "decidability" of the

theoretical framework. The "Hilbert Plan" finally failed because the "incompleteness theorem" of the young mathematician Godel in 1931 demonstrated that a formal theory including arithmetic could not guarantee its compatibility. Godel demonstrated in 1938 that the continuum hypothesis is consistent with the axioms of set theory, i.e., that the denial of the continuum hypothesis cannot be derived if the axioms of set theory are consistent. American mathematician Cohen established in 1963 that the continuum hypothesis cannot be inferred from a ZF system, i.e., the continuum hypothesis is undecidable in a ZF system. This implies that while the continuum hypothesis may hold in one axiomatic set theory, it might not hold in another. Just as the parallel axiom holds in Euclidean geometry, but not in non-Euclidean geometry, there are several different set theories in mathematics. It can also be concluded from Cohen's conclusion that not all mathematical problems can be proved or disproved, because in mathematics there are also undecidable propositions in addition to true and false propositions. Through the axiomatization of set theory, the threat of paradoxes to mathematics was indeed reduced, and mathematicians finally gained some sense of security, but this also weakened people's interest in the fundamentals of mathematics to a certain extent.

4. Conclusion

Set theory covers the basic mathematical concepts of sets, elements, and membership. In the majority of modern mathematics, set theory offers the vocabulary for describing mathematical objects. The axiomatic foundation of mathematics is provided through the use of undefines like "set" and "set member" in set theory, logic, and first-order logic. Naïve set theory views a set as a self-proving concept, similar to a collection. Mathematics' axiomatic set theory indirectly defines sets and set members using axioms that describe them. This notion treats sets and set members as points and lines, like Euclidean geometry. German mathematician Cantor, who flourished in the 19th and 20th centuries, is credited with creating set theory. A mathematician who influenced and divided. His late-19th-century work on continuity and infinity was contentious since it used infinity differently from mathematics. However, mathematics proved Cantor right. Many consider his set theory the most significant mathematical development of the 20th century. Sets have greatly expanded mathematical research and prepared for formalization. Modern philosophy and logic have been greatly influenced by set theory. In the years 1872 to 1893, he worked on set theory. He presented the idea of an infinite set, demonstrated the real number set's uncountability, advanced the "superfine number" theory, and put forth the "continuum hypothesis"—the idea that there isn't a base in between the natural number set and the real number set—as well as the concept of an infinite set. Set equivalence was established by Cantor to extend the concept of finite set elements to infinite sets. A straight line translates precisely into normal N-dimensional space, as shown by proof. He remarked, "I found it, but I couldn't believe it," shocked by the result. Cantor fully revealed infinite, reorganized mathematics, and promoted new mathematical subfields. When mathematicians worldwide celebrated the solidification of a mathematical foundation. Axiomatic set theory was developed by German mathematician Zermelo as a strategy for addressing and resolving the paradoxes that are intrinsic to the subject of mathematics. Taking this technique necessitates making use of the axiomatic method while set theory provides the underlying framework. The person argues that the paradox occurs because Cantor failed to lay limitations on the idea of a set, and the person also argues that Cantor's notion of a set is unclear. Both of these arguments are made in the context of the paradox. Both of these contentions are founded on the supposition that Cantor did not adequately limit the applicability of the idea of a set in his work. Zermelo correctly predicted that the shortened form of the axioms would make it simpler to define a set and comment on the traits it has, and his hypothesis was shown to be right. The first of the main objectives of this study is the construction of an axiomatization for set theory. The study of the similarities and contrasts between diverse approaches to the development of set theory is the second component of this undertaking. Even though there are fascinating puzzles within the universe of set theory that have not yet been solved, the axiomatization of set theory is largely regarded as a great triumph in the field of study.

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