Analysis of the Residue Theorem and Its Applications

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Abstract. Residue theorem is a useful tool for studying complex variable function. Residue theorem can be widely used in various disciplines, and it helps people solve a lot of practical problems. The residue theorem plays an important role in both mathematics and physics. It can not only make integral easier, but also solve problems that related to thermodynamics, optics, and electricity. In practice, it is usually complicated to solve the definite integral by using substitution, partial integration method and so on. Although these methods can calculate the solution, the computational workload is too large and complex. Using residue theorem to transform integral formulas can help people solve integrals easily. The method of using the residue theorem to solve the integral is more convenient than some traditional methods. In this paper, some applications of residue theorem are introduced. Further, it also gives some examples for calculating the integral by using residue theorem and logarithmic residue theorem.

Keywords: Residue theorem; Definite integral; Logarithmic residue; Complex analysis.

1. Introduction

Residue theorem is a very important tool in studying complex variable functions and calculus. In particular, it is convenient for people to solve the integral problem with singular point. The residue theorem is a generalization of the Cauchy integral theorem and Cauchy integral formula. In 1825, a French scientist Cauchy gave the definition of residue [1]. Subsequently, Cauchy further developed and refined the concept of inflow, after which he formed Cauchy residue theorem.

The residue theorem is a very useful tool for calculating the path integral of an analytic function along a close curve. Zhou et al. computed integrals $\int_0^{+\infty}\frac{dx}{1+x^n}$ and $\int_0^{+\infty}\frac{dx}{1-x^n}$ in two different ways [2]. The first method is summarized in Ref. [3], which divided integrals into two cases of non-singular points and singular points on the real axis, and then computed infinite integrals. The second method uses multivalued function method to solve integrals. Base on the integrating variables which make $y = x^n$, the real integral $I = \int_0^{+\infty}\frac{dx}{1+x^n}$ is turned into $I = \frac{1}{n}\int_0^{+\infty}\frac{y^{1/n-1}dy}{1+y}$. The function being integrated can be chosen as $f(z) = \frac{z^{1/n-1}}{1+z}$, which is a multivalued function. These researchers found that when people use the multivalued function method, the integral can be calculated as long as n is a real number. Further, Li et al. gave a generalization of a single residue theorem [4]. It is stated that if C is a simple closed curve in clockwise and it is analytic on the whole complex plane except for a finite number of points inside C, then

$$\int_C f(z)dz = 2\pi i Res_{z=0}\left[\frac{1}{z^{k+1}}f\left(\frac{1}{z^k}\right)\right]\ k = 1,2,3,\dots. \tag{1}$$

This theorem is very useful to compute the integral of single singular point. Zhang et al. gave people a generalization of the logarithmic residue base on the traditional logarithmic theorem, and gave many examples of application.

In this paper, the author will introduce how to use Taylor series and Laurent series to expand a function, and will compute the integral by Cauchy residue theorem in the first example in this article. It can help people better understand and apply the residue theorem and Laurent series in the calculation of integrals. Secondly, this paper introduces how to solve the integral in the form of $\int_{-\infty}^{+\infty} f(x)dx$ in second example. Thirdly, the author will show how to use the residue theorem to compute it when other theorems are difficult to do so. Fourthly, this paper will introduce how to use residue theorem to find the limit point of an analytic function and compute the logarithmic residue $\frac{1}{2\pi i}\int_{c} \frac{f'(Z)}{f(Z)}dz$. The author summarized these theories and examples in fourth part.

2. Basic Theorems

2.1. Residue theorem

Suppose that a function f is an analytic function in an annular domain $R_1 < |z - z_0| < R_2$ and the center of the domain is z_0, then it is supposed that C indicates any positively oriented simple closed contour around z_0 and they are located in that domain. Thus, at each point in the domain, $f(z)$ has the series representation

$$f(z) = \sum_{n=1}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \ (R_1 < |z - z_0| < R_2). \tag{2}$$

Suppose that $f(z)$ has an isolated point z_0 in the neighborhood $0 < |z - z_0| < \delta$ of z_0 [5], so analytic function $f(z)$ can expand by a Laurent series into

$$f(z) = \sum_{n=\infty}^{+\infty} C_n(z - z_0)^n. \tag{3}$$

Since $f(z)$ exists in the ring neighborhood U and C is a closed curve in the ring neighborhood U, then it is direct to find that

$$\oint_{c} f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}[f(z), z_k]. \tag{4}$$

Here, $Res[f(z), z_0]$ is the residue at z_0, and this value is equal to the coefficient of a Laurent series to the negative first power in the ring neighborhood U. So, Res $[f(z), z_0] = C^{-1}$ [6]. Suppose a function f is analytic in a simple closed contour C in the positive sense, except for a finite number z_k $(k = 1,2, \dots, n)$ inside C, then it is observed that Eq. (4) holds.

2.2. Logarithmic Residue

If function $f(z)$ can be analyzed on a simple closed curve C and the value of this function is not zero, and it is resolved in C everywhere except for a finite number of limits point inside C. Thus, the logarithmic residue is equal to the difference between the number of zero point and limit point inside a simple closed curve C. The logarithmic residue of function $f(z)$ for closed curve C is the following form.

Lemma: Let the function resolves on a close curve and the solution not be zero, and it is resolved in C everywhere except for a finite number of limit point inside C [7]. Therefore,

$$\frac{1}{2\pi i}\int_{c} \frac{f'^{(z)}dz}{f(z)}dz = N(f, C) - P(f, C). \tag{5}$$

Theorem: Let the function f has solution and it is not zero on the closed curve C, and $f(z)$ is analytic inside the closed curve C except for limit points that possible in C. $\delta(z)$ has analysis within and on C. $a_k(k = 1,2, \dots, p)$ are different zero points within C, its series corresponds to n_k. In addition, b_j $(j = 1,2, \dots, q)$ are different limit points within C and its series corresponds to m_j. Therefore, it is found that

$$\frac{1}{2\pi i}\int_{c}\frac{f'^{(z)}\delta(z)}{f(z)}dz=\sum_{k=1}^{p}n_k\delta(a_k)-\sum_{j=1}^{q}m_j\delta(b_j). \quad (6)$$

2.3. Cauchy's Integral Formula

Let z_0 be any point interior to a simple closed contour C, which is taken in the positive sense. The function f is analytic everywhere in contour C, then

$$f(z_0)=\frac{1}{2\pi i}\int_{c}\frac{f(z)dz}{z-z_0}. \quad (7)$$

3. Application of the Residue Theorem

The first example is based on the function $f(z)=\frac{6z^3+(6i-2)z^2+10z-6i-2}{z^4-1}$. The problem is to evaluate the integral

$$I=\int\delta f(z)dz \quad (8)$$

in which $\delta(t)=\frac{i-1}{2}+\sqrt{2}e^{it}$ and $0\le t\le 2\pi$.

To begin with, based on the Residue theorem shown in Eq. (4), the task is to find the residue on point z_k. There are four roots of $z=z_k$. If $z=z_k$, the formula is not valid. So z_k equals to $i,-i,1,-1$. By finding the value of these residue, the result of the integral can be calculated.

It is clear that Res $[f(z);z_0]$ is equal to the coefficient of a Laurent series to the negative first power in the ring neighborhood U, so the formula is expanded into Laurent series. Assume $z^4-1=h(z), 6z^3+(6i-2)z^2+10z-6i-2=g(z)$, then

$$g(z_k)\times\frac{1}{h(z_k)}=\frac{zk\cdot g(z_k)}{4}\cdot\frac{1}{z-z_k}+\left((z-zk)+\cdots\right). \quad (9)$$

Therefore, $\frac{zk\cdot g(zk)}{4}$ is the residue at $z=z_0$. Note that $Res(f;-1)=5$, $Res(f;1)=3, Res(f;\ i)=2$, $Res(f;-i)=-4$. According to the geometric meaning of the grinding length in the complex plane, it is found that $\left|z-\frac{i-1}{2}\right|=\left|\sqrt{2}e^{it}\right|=\sqrt{2}$, as $\sqrt{2}$ is radius of a circle centered on $\frac{i-1}{2}$. It is found that $Res(f,i)=2$ and $Res(f,-1)=5$ are in this neighborhood, see Fig. 1. So, the Residue theorem tells that

$$\int\delta f(z)dz=2\pi i(2+5)=14\pi i. \quad (10)$$

Note that if function is holomorphic in complex field C except $z_1,z_2,\cdots,z_n$, so the residue of $z=\infty$ equal to zero.

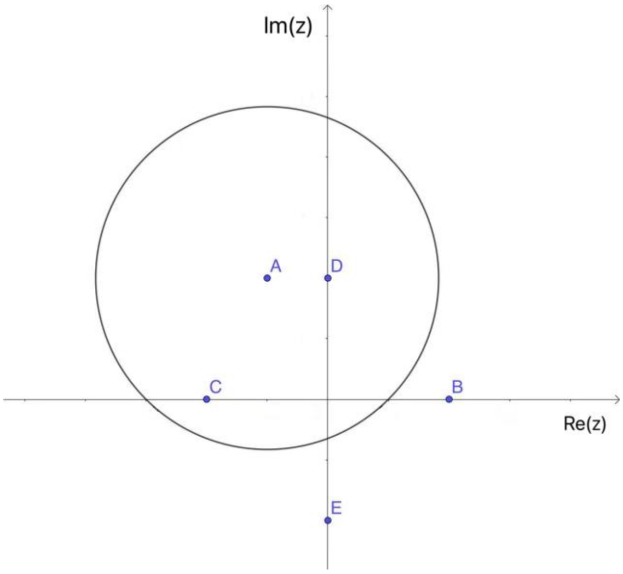

Fig. 1 The neighborhood of the integral in example 1

The second example is to compute the integral $I = \int_0^{+\infty} f(z)\,dx$, in which

$$f(z) = \frac{x^2}{x^4+6x^2+13} \tag{11}$$

The author firstly constructs a semicircular perimeter as shown in Fig. 2. It is found that there are four roots $\pm\sqrt{-3-2i}$, $\pm\sqrt{-3+2i}$ in the fraction $\frac{x^2}{x^4+6x^2+13}$, where the denominator does not exist at these values. Because this semicircle is in the upper half plane, so it is found that $z_1 = \sqrt{-3+2i}$ and $z_2 = \sqrt{-3+2i}$. Then the author can calculate the residue $\mathrm{Res}[f(z); \sqrt{-3+2i}] = \lim_{x\to z_1} \frac{z^2(z-\sqrt{-3+2i})}{z^4+6z^2+13} = \frac{1}{8}\sqrt{3-2i}$, and $Res[f(z); \sqrt{-3-2i}\ ] = \lim_{x\to z_2} \frac{z^2(z+\sqrt{-3-2i})}{z^4+6z^2+13} = -\frac{1}{8}\sqrt{3+2i}$. Therefore, the integral is

$$I = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{x^2}{x^4+6x^2+13}\,dx = \pi i\left(-\frac{1}{8}\sqrt{3+2i} + \frac{1}{8}\sqrt{3-2i}\right) = \frac{\pi}{4}\sqrt{\frac{\sqrt{13}-3}{2}}. \tag{12}$$

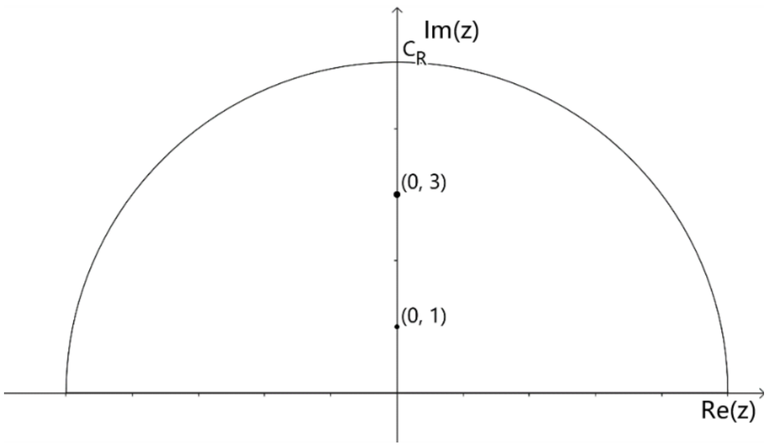

Fig. 2 The contour of the integral in example 2

The third example is to compute the integral [8]

$$I = \int_{|z|=n} \tan \pi z\, dz. \tag{13}$$

The integrand function is $\tan\pi z = \frac{\sin\pi}{\cos\pi}$, and its first order limit is $z = k + \frac{1}{2}$ $(k = 0, 1, -1 \ldots)$. It can find that $\mathrm{Res}(\tan\pi z)_{k+1/2} = \frac{\sin\pi z}{(\cos\pi z)'}\Big|x = k + \frac{1}{2} = -\frac{1}{\pi}$ (k=0, 1, -1...). Thus, based on Cauchy residue theorem, it is arrived that [9]

$$\int_{|z|=n} \tan \pi z \, dz = 2\pi i \sum_{|k+1/2|<n} \mathrm{Res}(\tan\pi z)_{z=k+\frac{1}{2}} = -4ni. \tag{14}$$

Finally, the fourth example is to compute the integral [10]

$$I = \int_{|z|} z^n \tan(\pi z^n) \, dz. \tag{15}$$

To solve it, it is useful to introduce the identity $\tan(\pi z^n) = \frac{sin(\pi z^n)}{cos(\pi z^n)}$. Let the function $f(z) = \frac{1}{\cos(\pi z)}$, so $\frac{f'(z)}{f(Z)} = \frac{-\pi \sin(\pi z)}{\cos(\pi z)}$. The function $f(z)$ has two first-order poles in $|z| = 1$, $\frac{1}{2}$ and $-\frac{1}{2}$. So based on the lemma shown earlily, it can be found that

$$I = \int_{[z]} \left(\frac{-\pi\sin(\pi z)}{\cos(\pi z)}\right)\left(\frac{z^n}{\pi}\right) dz = -2i\left[\left(\frac{1}{2}\right)^n + \left(-\frac{1}{2}\right)^n\right]. \tag{16}$$

4. Conclusion

In this paper, the author summarizes the various applications of the residue theorem and gives the applications of residue theorem in contour integrals and logarithmic integrals. To this end, the author gives four examples. The first one is expanding denominator and nominator of the function in the form of Taylor series, and then finding the integral by taking its residues and using residue theorem. The second one is by building a contour to solve the integral in the specific form. The third example is solving problems which are difficult to calculate by using the Cauchy integral formula. The fourth example is a generalization of residue theorem. the author uses the logarithmic residue and gives an example about the application of logarithmic residue theorem. This theorem can be used to solve many complex integrals. The author introduces some backgrounds of the residue theorem and Cauchy integral formula before the research was finished. It can help people understand the theorem and use these theorems to solve many extensive questions. To summarize, the residue theorem is a generalization of Cauchy integral theorem and Cauchy integral formula. Residue represents a special value of a complex function at some singular points. Residue theorem can help people solve many difficult integral problems. It can turn complex integrals into algebraic computations. It is expected that there will be more research results with the help of the residue theorem.

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