The Common Techniques and Applications in Calculus

Jiaxuan Qian*

Shanghai Yan'an High School, Shanghai, China

*Corresponding author: 2904313Q@student.gla.ac.uk

Abstract. Calculus is a pivotal subject in advanced mathematics since it is indispensable in various areas nowadays. It has many applications in other subjects like physics engineering and social science. However, compared with elementary mathematics, calculus is not easy to learn and integral is not easy to calculate because it is usually challenging to find out the antiderivative or the integral does not have a closed form. Thus, it is necessary to learn at least the basic techniques in calculus, which can help simplify the difficulty to solve the integrals in some extent. In this paper, the author will introduce three common techniques that everyone who learns calculus may use. These techniques are substitution, partial fraction, and integration by part, and illustrate their applications on several representative examples. In doing so, this can help readers grasp the most fundamental techniques in calculus, which is beneficial for them to further learn the calculus and apply them skillfully to calculate various integrals.

Keywords: Calculus; Definite integral; Integration by part; Variable substitution.

1. Introduction

Calculus was developed in the late 17th century by Isaac Newton and Gottfried Wilhelm Leibniz. It is not only an important part in mathematics, but widely used in science, engineering, and social science. Although modern calculus was developed in the last four century, the rudiment and thought of calculus has been used in the long history. In 1820 BC, Egyptian could calculate some area and volume by single variable calculus. However, no one nowadays knows how they got the brilliant formula [1,2]. In ancient Greek, Archimedes solved some complicated problems like calculating the center of gravity of a solid hemisphere, the center of gravity of a frustum of a circular paraboloid, and the area of a region bounded by a parabola and one of its secant lines [3,4].

Nowadays, calculus plays an important role in various areas. In complex analysis, grasping the fundamental thought of definite integral could simplify calculating infinite calculus in trigonal functions [5]. Using Sommerfeld integral by Chebyshev expansion can simplify the calculation in electromagnetic field, which extends the application in Physics [6]. Integral of Bessel function of arbitrary order increases the accuracy to calculate electric potential and field, solving the drawback in Filon-type methods [7]. In analysis, the improper integral has many applications in the cutting-edge technology. Laplace Transform can be used in circuit analysis, solving many complex improper integral. Dirichlet integral helps study damped vibration. Euler-Poisson integral has important application in probability theory [8]. In Newtonian mechanics, Fractional Calculus helps standard mechanics and denoises the reflection component to reduce and eliminate noise interference, which is the foundation of fractional mechanics [9]. Hence, calculus is a very natural extension of the elementary mathematics and in various areas, human being are used to using it to solve many complicated problems. No matter in the pure mathematics or other science fields even the arts, people need to use calculus [10]. So, it is important to learn the techniques to solve the calculus.
In this paper, the author will introduce some simple and basic techniques that everyone uses them when solving the calculus. Then, the author will introduce some applications to illustrate how to use these techniques properly.

2. Simple Techniques

Unlike taking derivative, calculations of integrals are by no means an easy task. Hence, the author will introduce some common techniques in this section.

2.1. Variable Substitution

Substitution is an important technique to calculate integrals since in the reality, there are various complicated integrals and it is natural to simplify them to calculate. The general form of the substitution can be expressed as [3]

$$\int u'(x)f\big(u(x)\big)dx = F\big(u(x)\big) + C\,. \tag{1}$$

It is easy to find that the derivative of the right-hand side is the left-hand side by using the chain rule.

Besides some simple substitution, trigonal substitution is often used in calculus. In some cases, like $\mathrm{u} = \sin x$, $\mathrm{u} = \cos x$, and $\mathrm{u} = \tan x$ can solve some complicated integrals quickly. Now, three common derivatives of trigonal functions are introduced to further illustrate the connection between trigonal function and other elementary functions, which are

$$(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}, (\cos^{-1} x)' = \frac{-1}{\sqrt{1-x^2}}\ (\tan^{-1} x)' = \frac{1}{1+x^2}. \tag{2}$$

Thus, when doing calculus, it is natural to connect trigonal substitution with formula like $\frac{1}{\sqrt{1-x^2}}$.

For the benefit of readers, some simple proof of these three formulas in Eq. (2) are presented. For simplicity, the author will take $(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$ for instance. Suppose that there is a point $A\,(x\,, \sin x)$ on $y = \sin x$, then the derivative of the point is $\cos x$. According to the property of inverse function, there is a point $(\sin x, x)$ on $\sin^{-1} x$ and the derivative on this point is $\frac{1}{\cos x}$. Now, suppose $t = \sin x$ and $\frac{1}{cosx} = \frac{1}{\sqrt{1-t^2}}$. Hence, $\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$. Use the same way, one can find the derivative of $\cos^{-1} x$. Since the derivative of $\sin^{-1} x$ and $\cos^{-1} x$ have been found, it is easy to find $\tan^{-1} x$. Suppose $y = \tan x$, then $\frac{dy}{dx} = \frac{1}{cos^2 x}$ and $\frac{dx}{dy} = cos^2 x = \frac{1}{1+y^2}$. Hence, the derivative is $(\tan^{-1} x)' = \frac{1}{1+x^2}$.

Now, some antiderivatives can be found by substitution. For example, for the integral $\int \frac{1}{(1+x^2)^2} dx$, it is nature to set $x = \tan\theta$ and thus $dx = \frac{1}{\cos^2\theta} d\theta$. Now, the integral can be written as $\int \cos^2\theta d\theta = \frac{\sin 2\theta}{4} + \frac{\theta}{2} + C = \frac{1}{2}\tan^{-1} x + \frac{x}{1+x^2} + C$. Another example is $\int \frac{1}{x\sqrt{1-x^2}} dx$. Suppose that $x = \sin\theta$, then $dx = \cos\theta d\theta$. Now the integral can be written as $\int \frac{1}{\sin\theta} d\theta$. Next, let $u = cos\theta$, then $du = -\sin\theta\, d\theta$. Thus, $\int \frac{1}{\sin\theta} d\theta = -\int \frac{1}{1-u^2} du = \frac{1}{2}\ln\left|1 - \sqrt{1-x^2}\right| - \frac{1}{2}\ln\left|\sqrt{1-x^2} + 1\right| + C$. From the two examples, it is easy to find that when doing trigonal substitution, there are some common substitutions, since there are some connections between the derivative of $\sin x \cos x\ \tan x$ and some elementary functions.

2.2. Partial Fractions and Integration by Part

It is a common way to solve some fractions in calculus by using partial fractions. In this paper, the author will just introduce some simple functions to avoid calculating in matrix since it is more important to learn the essence. For example, it is easy to find that the integral identity $\int \frac{x-4}{x^2-5x+6} dx = \int \frac{2}{x-2} - \frac{1}{x-3} dx = 2\ln|x-2| - \ln|x-3| + C$ holds. The other method is the integration by part, which states that

$$\int u dv = uv - \int v du\,. \tag{3}$$

It is nearly the most important technique in calculus. By this technique, many challenging antiderivatives can be found. For example, the integral identity $\int \ln(1+x^2)dx = x\ln(1+x^2) - \int x\ln(1+x^2)'dx + C = x\ln(1+x^2) - 2x + 2\tan^{-1}x + C$ holds.

2.3. More Complicated Examples

To demonstrate the efficiency of these techniques, an example $\int \frac{dx}{x\sqrt{x^2+1}}$ is showed. By looking at the integral, there exists $\sqrt{x^2+1}$ in the denominator, so substitution can be tried. Suppose $x = \tan\theta$, then $dx = \frac{1}{cos^2\theta}d\theta$ and the integral can be written as $\int \frac{cos^2\theta}{\sin\theta}d\theta$. Compared with the original integral, this integral becomes much easier since in trigonometry there has more ways to compute the integral compared with a more complicated root. Now, to compute $\int \frac{cos^2\theta}{\sin\theta}d\theta$, use the trigonal substitution. Suppose $u = cos\theta$, then $du = -\sin\theta d\theta$. Thus, the integral can be simplified as $\int \left(\frac{1}{1-u^2} - 1\right)du$. Finally, the result is $\int \frac{dx}{x\sqrt{x^2+1}} = \frac{1}{2}\ln\left|1 - \frac{1}{\sqrt{1+x^2}}\right| + \frac{1}{2}\ln\left|1 + \frac{1}{\sqrt{1+x^2}}\right| - \frac{1}{\sqrt{1+x^2}} + C$.

3. Examples and Applications

3.1. Example 1

Compute the integral [5]

$$I = \int_1^{e^{\frac{1}{3}}} \frac{x^2\ln x}{(1+x^3)^2}dx. \tag{4}$$

This integral seems to be complicated and some substitutions should be used to simplify it. First, suppose $\mathrm{u} = \mathrm{x}^3$, then $\mathrm{du} = 3\mathrm{x}^2\mathrm{dx}$ and $\ln \mathrm{x} = \frac{1}{3}\ln \mathrm{u}$. The integral can be simplified as

$$I = \int_1^e \frac{\ln u}{(1+u)^2}du = -\frac{1}{9}\frac{\ln u}{1+u}\bigg|_1^e + \int_1^e \frac{1}{u(1+u)}du\,, \tag{5}$$

where the integration by part is also used, making the integral much cleaner. Note that the integral contains function like $\ln x$, whose antiderivative is not easy to find. The second part of the integral in Eq. (5) can be handled by the partial fraction, and the integral can be simplified as

$$I = -\frac{1}{9}\frac{1}{(1+e)} + \int_1^e \frac{1}{u} - \int_1^e \frac{1}{1+u}. \tag{6}$$

Now, it is simple to come to the conclusion that

$$I = \int_1^{e^{\frac{1}{3}}} \frac{x^2\ln x}{(1+x^3)^2}dx = 1 + \ln 2 - \frac{1}{9}\frac{1}{(1+e)} - \ln(1+e). \tag{7}$$

3.2. Example 2

Compute the integral of [6]

$$I = \int_0^1 \frac{dx}{(1+x)^{\frac{1}{3}}(1-x)^{\frac{2}{3}}}. \tag{8}$$

The integral is a common polynomial in the fraction. So, first, it is natural to rationalize the denominator. Multiplying $(1+x)^{\frac{1}{3}}$ in both the numerator and the denominator, the integral becomes to be

$$I=\int_0^1 \frac{(1+x)^{\frac{1}{3}}}{(1+x)^{\frac{2}{3}}(1-x)^{\frac{2}{3}}}dx=\int_0^1 \frac{(1+x)^{\frac{1}{3}}}{(1-x^2)^{\frac{2}{3}}}dx. \quad (9)$$

Through the technique of substitution, suppose $x=\cos\theta$, then $dx=-\sin\theta\, d\theta$ and thus the integral can be further simplified as

$$I\ =\int_0^{\frac{\pi}{2}} \frac{(1+\cos\theta\,,)^{\frac{1}{3}}sin\theta}{\left((sin^2)\theta\right)^{\frac{2}{3}}}d\theta=\int_0^{\frac{\pi}{2}}\left(\cot^{\frac{1}{3}}\frac{\theta}{2}\right)d\theta. \quad (10)$$

When computing an integral, it is useful to turn multiple variables to a single variable. So, through simplifying the variables to $\theta/2$ the integral becomes much cleaner. The variables in this integral are replaced by one integral. The power of 1/3 seems tricky here, so it is wise to substitute it. Hence, suppose $u^3=\cot\frac{\theta}{2}$, then $d\theta=\frac{-3u^2}{2(1+u^6)}du$. Therefore, the function can be simplified as

$$I=\frac{3}{2}\int_0^{\infty}\frac{u^3}{1+u^6}du\,. \quad (11)$$

To make the integral cleaner, suppose $t=u^3$, then $dt=3u^2du$. Thus, the integral is simplified as

$$I=\frac{3}{4}\int_0^{\infty}\frac{t}{(t+1)(t^2-t+1)}dt. \quad (12)$$

By using the partial fraction, it is further simplified as

$$I=-\frac{1}{6}\ln(t+1)|\begin{matrix}\infty\\0\end{matrix}+\frac{1}{6}\int_0^{\infty}\frac{t+1}{t^2-t+1}dt\,. \quad (13)$$

To proceed further, the second part of the function need to be changed in order to get the integral, i.e., $\frac{t+1}{t^2-t+1}=\frac{2t-1}{2(t^2-t+1)}+\frac{3}{2(t^2-t+1)}$. It is easy to find the integral of the first part of the function, which is $\frac{1}{2}\ln(t^2-t+1)$ since the numerator is the derivative of t^2-t+1. Now, the aim is to find the integral of the second part. First, compared with the trigonal substitution, $(\tan^{-1}x)'=\frac{1}{1+x^2}$. So, if the denominator only has $1+x^2$, the integral can be got. Hence, one can write the second part as$\frac{3}{2(x^2-x+1)}=\sqrt{3}\frac{1}{\left(\left(\frac{2x-1}{\sqrt{3}}\right)^2+1\right)}\frac{2}{\sqrt{3}}$. ow, try the substitution $\tan\alpha=\frac{2x-1}{3}$ and simplify it, which is $\sqrt{3}\,(\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right))$. Hence,

$$I=-\frac{1}{6}\ln(t+1)\Big|_0^{\infty}+\frac{1}{12}\ln(t^2-t+1)\Big|_0^{\infty}+\frac{\sqrt{3}}{6}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)\Big|_0^{\infty}=\frac{\sqrt{3}}{9}\pi. \quad (14)$$

3.3. Example 3

Compute the integral of [7]

$$I=\int_0^{\frac{\pi}{2}}\frac{\sin\left(\frac{2}{3}x\right)}{tanx}dx\,. \quad (15)$$

First, suppose $u=\frac{2}{3}x$, then $du=\frac{2}{3}dx$. Therefore, the integral can be written as

$$I = \frac{3}{2}\int_0^{\frac{\pi}{3}} \frac{\sin u}{\tan\frac{3}{2}u} du. \tag{16}$$

By doing this, $\sin u$ and $\tan\frac{3}{2}u$ can be replaced by $\tan\frac{x}{2}$. It is the thought of changing multiple variables to single variable. Since $\sin x = \frac{2\tan\frac{x}{2}}{1+tan^2\frac{x}{2}}$ and $\tan\frac{3}{2}x = \frac{3\tan x - tan^3 x}{1-3tan^2 x}$, and let $t = \tan\frac{u}{2}$, then the integral can be written as

$$I = \frac{3}{2}\int_0^{\frac{1}{\sqrt{3}}} \frac{\frac{2t}{1+t^2}}{\frac{3t-t^3}{1-3t^2}} \frac{2}{1+t^2} dt = 6\int_0^{\frac{1}{\sqrt{3}}} \frac{1-3t^2}{(3-t^2)(1+t^2)^2} dt. \tag{17}$$

Through these substitutions, the integral becomes a function like Eq. (8). However, the difference between the two functions is that the denominator in the new function is the power of the polynomial is integer. Hence, the substitution does not need to be used. Instead, using the partial fraction can solve it more conveniently. Hence, do the partial fraction of this integral. Then, the integral can be written as

$$I = 6\int_0^{\frac{1}{\sqrt{3}}} \frac{1}{4\sqrt{3}(t-\sqrt{3})} - \frac{1}{4\sqrt{3}(t+\sqrt{3})} + \frac{1}{(1+t^2)^2} - \frac{1}{2(1+t^2)} dt. \tag{18}$$

Hence,

$$I = \frac{\sqrt{3}}{2}\ln|t-\sqrt{3}|\Big|_0^{\frac{\sqrt{3}}{3}} - \frac{\sqrt{3}}{2}\ln|t+\sqrt{3}|\Big|_0^{\frac{\sqrt{3}}{3}} - \frac{1}{2}\tan^{-1} t\Big|_0^{\frac{\sqrt{3}}{3}} + \frac{1}{2}\left(\tan^{-1} x + \frac{x}{1+x^2}\right)\Big|_0^{\frac{\sqrt{3}}{3}}. \tag{19}$$

where the third part in Eq. (18) can be solved by using the substitution $t = \tan\theta$. By simplifying Eq. (19), it is found that

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{2}{3}x\right)}{tanx} dx = \frac{3\sqrt{3}}{4} - \frac{\sqrt{3}}{2}\ln 2. \tag{20}$$

3.4. Example 4

Compute the integral

$$I = \int_0^{\infty} e^{-ax} cosbxdx \ (a > 0). \tag{21}$$

This integral contains exponential and trigonometric function. However, it seems that substitution and partial fraction cannot be useful. Hence, the only way to do is to try integration by part, and the exponential function is convenient when using this technique since the derivative of the exponential function always contains itself. Hence, by using integration by part [10], it is

$$I = \int_0^{\infty} e^{-ax} cosbxdx = \frac{-1}{a}(e^{-ax}cosbx)\Big|_0^{\infty} - \frac{b}{a}\int_0^{\infty} e^{-ax} sinbxdx. \tag{22}$$

Looking at the formula and compared with the original integral, the second part of the new formula just changes from $cosx$ to $sinx$. It is natural to think to try again integration by part. According to the previous experience, it is not difficult to imagine that the formula can contain the original integral. Hence, doing integration by part again in the second part, the integral becomes to

$$I = \frac{-1}{a}(e^{-ax}cosbx)|\begin{matrix}\infty\\0\end{matrix} + \frac{b}{a^2}e^{-ax}sinbx\Big|\begin{matrix}\infty\\0\end{matrix} - \frac{b^2}{a^2}\int_0^{\infty} e^{-ax}cosbxdx\,. \qquad (23)$$

Here, $\int_0^{\infty} e^{-ax}cosbxdx$ comes back when doing the integral by part. Hence,

$$I = \int_0^{\infty} e^{-ax}cosbxdx = \frac{-1}{a}(e^{-ax}cosbx)|\begin{matrix}\infty\\0\end{matrix} + \frac{b}{a^2}e^{-ax}sinbx\Big|\begin{matrix}\infty\\0\end{matrix} - \frac{b^2}{a^2}\int_0^{\infty} e^{-ax}cosbxdx. \qquad (24)$$

Just by moving the third part in the right hand side to the left and divide $\frac{b^2}{a^2} + 1$. The conclusion comes out to be

$$I = \int_0^{\infty} e^{-ax}cosbxdx = \frac{a^2}{b^2 + a^2}\left(\frac{-1}{a}(e^{-ax}cosbx)|\begin{matrix}\infty\\0\end{matrix} + \frac{b}{a^2}e^{-ax}sinbx\Big|\begin{matrix}\infty\\0\end{matrix}\right). \qquad (25)$$

By simplifying the formula in the right-hand side, one finds that

$$I = \int_0^{\infty} e^{-ax}cosbxdx = \frac{a}{b^2 + a^2}\ (a > 0). \qquad (26)$$

Inspired by this calculation, readers can try to calculate $\int_0^{\infty} e^{-ax}sinbxdx\ (a > 0)$. Using the same way, one can come to the conclusion that

$$\int_0^{\infty} e^{-ax}sinbxdx = \frac{b}{b^2 + a^2}(a > 0). \qquad (27)$$

4. Conclusion

This paper introduces three common techniques in calculus and uses some representative examples to illustrate their applications. When calculating various integrals, variable substitution helps simplify the functions like trigonal function and function with root. The partial fractions can solve a kind of rational fractional functions, and integration by part helps find the integrals whose antiderivatives is hard to get like logarithmic function and function with various exponential functions and trigonal functions. After using the three techniques skillfully, readers may find it is easier to learn more advanced mathematics and can calculate various integrals in analysis. When using these techniques, the author also emphasizes some important thought like simplifying the complicated functions to a cleaner function and changing multiple variables to a single variable, which are more important for readers to learn mathematics in some extent since these are some basic ideas when studying mathematics or in other subjects. However, due to the limited space, the author only introduces the most basic ways to calculate integrals. In the advanced mathematics, there are many other ways to calculate integrals like Cauchy residue theorem, Cauchy integral formula and Fourier series. Readers can also learn these ways in paper about complex analysis, which is the extension of this paper.

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