Applications of Residue Theorem to Some Selected Integrals

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Abstract. A complex variable is a quantity that contains both a real part and an imaginary part. Complex analysis is an application of mathematics that analyzes the features of functions of complex variables. Physics, engineering, and computer science are just a few of the scientific disciplines that benefit from complex analysis. In order to tackle issues that are challenging or impossible to resolve using only real variables, complex analysis is crucial. This paper introduces an important theorem in complex analysis, which is Cauchy's residue theorem. A relative general expression for a complex integral along a simple closed contour is provided by Cauchy's residue theorem. Cauchy's residue theorem can be used to select an acceptable closed contour for the calculation of some unusual definite integrals that may be highly complex and challenging to solve using traditional methods. The residue at the function's isolated singularities is then established. In the formula that is subtracted in Cauchy's residue theorem, the values of the residues are extracted. The integral along a simple closed contour can then be represented in two pieces, in which one along the real axis and the other along the circle.

Keywords: Complex Analysis; Cauchy's residue theorem; Singularity; Analytic function.

1. Introduction

Cauchy's residue theorem makes a significant difference in the field of Complex Analysis, which allows the complex integral to be calculated by adding up the residues of the singularities in the complex plane. In order to establish the residue theorem, Cauchy took into account the difference of integrals along two pathways with conventional endpoints sandwiching a function pole in the middle. It is simple to see that the difference between the two path integrals may be represented as the loop integral of the closed loop created around the two paths by considering the minus sign as the direction of the path. If the function f is resolved in a concise closed profile C, apart from a limited number of singularities, these must all be isolated. The complex function is a comprehensive and systematic summary and induction of the characteristics and relationships of the mappings between complex number fields [1]. The Cauchy residue theorem indicates that the integral of the function f around C is given by the sum of the residues of f at the singular points inside C times $2\pi i$.

The residue theorem is still useful today for complicated analysis and has many applications in physics, math, and engineering. It has since been extended and generalized in various ways. For example, the theorem can be extended to functions defined on more general spaces, such as metric spaces or Banach spaces [2]. Overall, the residue theorem is a fundamental finding in mathematics that has a wide range of significant applications and extensions. Mathematicians are still actively working on it today. However, in the actual calculation process, there are many forms of integrable functions and the original function can not be integrated. The residue theorem in the function of complex variables provides an important theoretical calculation method for this kind of problem. For those functions of real variables that are difficult to solve analytically, people can use the residue theorem to solve them: the main idea is to transform the real variable function into a complex variable function, and then utilize the residue theorem to calculate and solve the integral.

In short, this paper is intended to generally summarize the simple definition of Cauchy's residue theorem and several other relevant knowledge points. Next, the author provides some applications of the residue theorem to four different integrals.

2. Method

2.1. Taylor Series and Laurent Series

The Taylor series is the function that represents the sum of infinite terms expressed by a function's derivatives at a single point. The Laurent series, which includes negative power words, has some similarities to it. The general equation of the Taylor series is

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{n}(x_0)}{n!}(x-x_0)^{n}. \tag{1}$$

Suppose that a function f is differentiable in an annular domain $R_1 < |z - z_0| < R_2$, centered at z_0. Laurent series represents functions in a form for the sum of infinite terms of derivatives in the function. It contains both infinite positive power terms and infinite negative power terms. In addition, in the infinite positive power terms, the coefficient of a_n is $\frac{1}{2\pi i}\oint_c f(z)\frac{1}{(z-z_0)^{n+1}}$. It can also be applied to find the residue and the integral of the complex function. The positive power term part and the negative power part are both present in the Laurent series' general equation

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^{n}+\sum_{n=0}^{\infty}b_n\frac{1}{(z-z_0)^{n}}, \tag{2}$$

where $R_1 < |z-z_0| < R_2$.

2.2. The Residue Theorem and Analytic Function

Cauchy's residue theorem is developed by Cauchy's integral formula

$$f(z_0)=\frac{1}{2\pi i}\oint_c\frac{f(z)}{z-z_0}dz \tag{3}$$

where f is a complex function that is defined analytical inside a closed and bounded contour C except for a finite number of singular points z_k $(k = 1,2,3 \dots \dots n)$ which is inside C. Let C be a simple closed, and positively oriented contour. Then the Cauchy residue theorem claims that the integral of the function f around C is given by the sum of the residues of f at the singular points inside C times $2\pi i$. Namely,

$$\int f(z)\,dz=2\pi i\sum Res\,[f(z), z=z_k] \tag{4}$$

Analytic complex functions are "differential" complex functions. Singularities are points or parts of some functions that are not analytical. A function is referred to as analytic or holomorphic if it is analytic throughout the entirety of its domain or if it contains any singularities.

2.3. Three types of singularities

The singularities simply mean the pots where $f(x)$ is not analytic. In the complex plane, there are mainly three types of singularities. They are called removable singularities, isolated singularities, and branch singularities.

A removable singularity is a point z_0 where the function $f(z_0)$ appears to be undefined but with the knowledge that $\lim_{z\to z_0} f(z)=\omega_0$ and the author assigns $f(z_0)$ the value ω_0. Then the author could say that he has "removed" this singularity, so the residue is equal to 0 for the removable singularities. For example, the point $z = 0$ would be a singularity of the function $f(z)=\frac{sin(z)}{z}$.

An isolated singularity is a point where the function is differentiable on the simple closed contour $0 < |z - z_0| < r$. However, this function is undefined at the point $z = z_0$. People always say isolated singularities poles. In other words, $f(z)$ has a singularity at z_0 that is not a limit point of other singularities of $f(z)$. Particularly, the residue of poles can be expressed in two different ways. If z_0 is a first-order pole, the residue could be expressed as

$$Res[f(z), z_0] = \lim_{z \to z_0} (z - z_0) f(z) \,. \tag{5}$$

If z_0 is a m-th order pole, the residue could be expressed as:

$$Res[f(z), z_0] = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m f(z) \,. \tag{6}$$

For example, the point $z = i$ is the isolated singularity of the function $f(z) = \frac{z}{z-i}$. A branch singularity is a point z_0 through which all possible branch cuts of a multi-valued function can be drawn to produce a single-valued function. An example of such a point would be the point $z = 0$ for $\log(z)$ [3].

There is also a special kind of isolated singularity, called an essential singularity. The canonical example of an essential singularity is $z = 0$ for the function $f(2) = e^{1/z}$. The easiest way to define an essential singularity of a function involves Laurent Series [4].

3. Applications of Residue Theorem

3.1. $\frac{P(x)}{Q(x)}$ – type Integral

The first integral that belongs to this type is [5]

$$I = \int_{\gamma} f(z)\, dz. \tag{7}$$

Let $f(z) = \frac{az^3+bz^2+cz+d}{z^4-1}$ with $a = 10, b = i + 1, c = -4, d = 1 - i$ and where $\gamma(t) = \frac{1-i}{2} + \sqrt{2}e^{it}, 0 \le t \le 2\pi$. This function has four singularities $1, -1, i, -i$ in the complex plane. In the $\gamma(t)$, there are only two singular points which are 1 and $-i$. Thus only the residues of these two points should be computed in this calculation. Firstly, the author should compute each Taylor expansion of the function's numerator and denominator. Then find the coefficient of the $\frac{1}{z-z_k}$ at the whole Taylor expansion of this function, which is $\frac{z_k(az_k^3+bz_k^2+cz^k+d)}{4}$. Hence two singular points are plugged in this coefficient to compute the residue at $\gamma(t)$, which is 2 at point 1 and 3 at point -i. According to the Cauchy residue theorem shown in Eq. (4), the author knows that the integral of this function are $2\pi i\,(2 + 3) = 10\pi i$.

The second integral that belongs to this type is [6]

$$I = \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2}\, dx. \tag{8}$$

Let $f(x) = \frac{1}{(1+x^2)^2}$ and it is clear that $f(x) \approx \frac{1}{z^4}$. By applying the residue theorem to the contour illustrated below, the author has

$$\int_{C_1 + C_R} f(z)\, dz = 2\pi i \sum Res[f(z), z = z_k] \,. \tag{9}$$

Each pieces in the above Eq. (9) are examined. For the $\int_{C_R} f(z)\, dz$, $\lim_{R\to\infty} \int_{C_R} f(z)\, dz = 0$. For the $\int_{C_1} f(z)\, dz$, it is directly to see that $\lim_{R\to\infty} \int_{C_1} f(z)\, dz = \lim_{R\to\infty} \int_{-R}^{R} f(x)\, dx = \int_{-\infty}^{\infty} f(x)\, dx = I$. Letting $R \to \infty$, Eq. (8) becomes

$$I = \int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum \text{Residues of } f \text{ inside the contour.} \tag{10}$$

In the end, the required residues are calculated. $f(z)$ has poles of order two at $\pm i$. Only $z = i$ is inside the contour and its residue is calculated. Let $g(z) = (z - i)^2 f(z) = \frac{1}{(1+i)^2}$, then the $Res(f, i) = g'(i) = -\frac{2}{(2i)^3} = \frac{1}{4i}$. Thus,

$$I = 2\pi i Res(f, i) = \frac{\pi}{2}. \tag{11}$$

3.2. $R(\cos\theta, \sin\theta)$-type Integral

Consider the following integral over the interval [0,2π]

$$I = \int_0^{2\pi} \frac{d\theta}{1 - 2p\cos\theta + p^2} \tag{12}$$

where $0 < p < 1$. The first step is to introduce a complex variable according to $z = e^{i\theta}$, so that $\cos\theta = \frac{1}{2}(z + \frac{1}{z})$ and $dz = ie^{i\theta}d\theta = izd\theta$. Therefore, the equation can be rewritten as an integral around a closed contour C which is a unit circle about the origin [7]

$$I = \frac{1}{i}\oint dz \frac{1}{(1 - pz)(z - p)} \tag{13}$$

The Eq. (13) has two singular points but just one singular point $z = p$ is in the contour C. Therefore, the Residue theorem implies that $I = \frac{1}{i} 2\pi i\, Res[f(z), z = \text{p}] = \frac{2\pi}{1-p^2}$. From the well-known Kepler's dilemma, the integral is equivalent to

$$\phi(\epsilon) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\sin\theta^2}{(1 + \epsilon\cos\theta)^2} d\theta = -\frac{1}{4\pi i}\oint_{|z|=1} \frac{(z^2 - 1)^2\, dz}{z[2z + \epsilon(z^2 + 1)]^2}. \tag{14}$$

The poles are at $z_0 = 0$ and when $2z + \epsilon(z^2 + 1) = 0$, it is found that the roots are $z_1 = -1 - \sqrt{1-\epsilon}2\,\epsilon$ and $z_2 = -1 + \sqrt{1-\epsilon}\,2\,\epsilon$. Obviously, z_0 lies into the unit circle. From the relation $1 + \sqrt{1 - \epsilon^2} > 1 > \epsilon$, it follows that $z_1 < -1$ and, consequently, $-1 < z_2 < 0$. Namely, z_2 is inside the unit disk, and z_1 is outside. Let $f(z) = -\frac{1}{2\epsilon^2}\frac{(z^2-1)^2}{z(z-z_1)^2(z-z_2)^2}$, then $\phi(\epsilon) = Res(f, 0) + (f, z_2)$. Straightforwardly, $Res(f, 0) = \lim_{z\to 0} zf(z) = -\frac{1}{2\epsilon^2}$. For the second-order pole at $z = z_2$, $Res(f, z2) = \lim_{z\to 0}\frac{d}{dz}((z - z_2)^2 f(z))$. Specifically, for $\epsilon = 0$ the integral becomes

$$\phi(0) = -\frac{1}{4\pi i}\oint_{|z|=1} \frac{(z^2 - 1)^2}{2z^3} dz = \frac{1}{2}. \tag{15}$$

3.3. $f(x)\ln x$-type and $f(x)/x^a$ -type Integral

As an illustration, the paper considers the integral [8]

$$I = \int_0^{\infty} \frac{\ln x}{(x + 1)^4} dx. \tag{16}$$

The integral can be calculated by the formula $\int_0^{\infty} f(x)\ln dx = -\frac{1}{2} Re \sum Res(f(z)\ln z^2$ that is derived from the Cauchy's residue theorem. Letting $f(x) = \frac{1}{(z+1)^4}$, it is arrived that

$$\int_0^{\infty}\frac{ln\,x}{(x+1)^4}dx=-\frac{1}{2}\text{Re}\left[Res\left[\frac{ln\,z^2}{(z+1)^4},z=-1\right]\right]=-\frac{1}{2}. \tag{17}$$

The second integral is [9]

$$I=\int_0^{\infty}\frac{d\text{x}}{\text{x}^4+\text{a}^4},a>0. \tag{18}$$

Letting $f(x)=\frac{1}{\text{x}^4+\text{a}^4}$, so that $\int_0^{+\infty}\frac{dx}{\text{x}^4+\text{a}^4}=\frac{1}{2}\int_{-\infty}^{+\infty}\frac{dx}{\text{x}^4+\text{a}^4}$. There are four singularities which are $\text{z}_\text{k}=a\text{e}^{\frac{\pi+2\text{k}\pi}{4}\text{i}},k=0,1,2,3$. Let $f(x)=\frac{Q(x)}{P(x)}$ be a rational formula, in the formula that $P(\text{z})=\text{c}_0\text{z}^\text{m}+\text{c}_1\text{z}^{\text{m}-1}+\cdots+\text{c}_\text{m},\text{c}_0\neq 0$ and $Q(\text{z})=\text{b}_0\text{z}^\text{n}+\text{b}_1\text{z}^{\text{n}-1}+\cdots+\text{b}_\text{n},\text{b}_0\neq 0$. These equations are prime polynomials and meet the conditions: $n-m\geq 2,Q(z)\neq 0$ on real axis. Therefore, it is found that

$$\int_0^{\infty}f(x)\,dx=2\pi i\sum_{\text{Imz}_\text{k}>0}\text{Res}\,f(\text{z}),z=\text{z}_\text{k} \tag{19}$$

Here, $Resf(x)=\frac{1}{4\text{z}^3}(z=\text{z}_\text{k})=\frac{1}{4{\text{z}_\text{k}}^3}=\frac{\text{z}_\text{k}}{4{\text{z}_\text{k}}^4}=-\frac{\text{z}_\text{k}}{4\text{z}^4},k=0{,}1{,}2{,}3$. $f(x)$ just has two singularities a_0 and a_1 in the upper half complex plane. Then

$$\int_0^{\infty}\frac{dx}{x^4+a^4}=-\pi i\frac{1}{4a^4}\left(ae^{\frac{\pi}{4}i}+ae^{\frac{3\pi}{4}i}\right)=-\pi i\frac{1}{4a^3}\left(e^{\frac{\pi}{4}i}-e^{\frac{\pi}{4}i}\right)=\frac{\pi}{2a^3}sin\frac{\pi}{4}=\frac{\pi}{2\sqrt{2}a^3}. \tag{20}$$

3.4. $f(x)e^{imx}$ -type integral

In this part, the example is [10]

$$I=\int_{-\infty}^{\infty}\frac{e^{itx}}{x^2+1}dx \tag{21}$$

From point $-a$ to point a, it travels in a true straight line before turning counterclockwise along a semicircle centered on point $-a$ to point a. In order for the hypothetical unit i to be contained within the curve, author takes a to be greater than 1. Thus the path integral are $\int_C f(z)\,dz=\int_C\frac{e^{itz}}{z^2+1}dz$. Since $f(z)$is an integral function, it only has singularities if $z^2+1=0$, and $z^2+1=(z+i)(z-i)$. Hence the singularities of $f(z)$ are $z=I$ and $z=-i$. Due to $f(z)$ has the following relation

$$\frac{e^{itz}}{z^2+1}=\frac{e^{itz}}{2i}\left(\frac{1}{z-i}-\frac{1}{z+i}\right)=\frac{e^{itz}}{2i}\frac{1}{z-i}-\frac{e^{itz}}{2i(z+i)}, \tag{22}$$

the residue of $f(z)$ at $z=i$ is $\frac{e^{-t}}{2i}$. According to the residue theorem, it can get that: $\int_C f(z)\,dz=2\pi iRes\left[f(z),z=i=2\pi i\frac{e^{-t}}{2i}=\pi e^{-t}\right]$ .The path C could be divided into $\int_{straight}f(x)+\int_{arc}f(x)=\pi e^{-t}$, and thus $\int_{-a}^{a}f(x)=\pi e^{-t}-\int_{arc}f(x)$. If $t>0$, the integral along the semicircle's path goes to zero as the semicircle's radius approaches infinity. Then $\int_{-\infty}^{\infty}\frac{e^{itz}}{z^2+1}dz=\pi e^{-t}$. Analogously, the author has $\int_{-\infty}^{\infty}\frac{e^{itz}}{z^2+1}dz=\pi e^{t}$. Thus, it is found that

$$\int_{-\infty}^{\infty}\frac{e^{itz}}{z^2+1}dz=\pi e^{-|t|}. \tag{23}$$

If $t=0$, the fundamental approach can be used to quickly get this integral's value, which is π.

4. Conclusion

In studying complex variable functions, the residues play a significant role. In this paper, the author discusses the different kinds of problems of residues in the complex plane. First, this paper introduces basic knowledge of the residue theorem and several relative mathematical points like Taylor series and Laurent series which could help calculate integrals. Then the paper provides several applications of the residue theorem to typical improper integrals. The author gives four types of integrals including the trigonometric function, rational expression, logarithmic function, and exponential function these four types of researched integrals have a wide range of uses in physics, chemistry, and economics, including the evaluation of market selling prices, supply and demand, and solutions to specific atomic issues. In addition, the residue theorem is a fundamental concept in the theory of complex variable functions. It offers an efficient method for resolving a variety of improper integral issues involving real variable integrands. With the help of the residue theorem, people could evaluate complex integrals by just identifying the singularities of the function in the complex plane and then utilizing the residue theorem. Thus the residue theorem provides a useful tool for resolving complex problems that may be difficult to approach using traditional approaches. With its assistance, integrals can be calculated in a simple and more understandable manner. In conclusion, this work is useful for the method and idea expansion of improper integral calculation and promotes the effective solution of integral calculation in real-world problems and it does this by using example problems.

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