The Residue Theorem: Applications in Complex Integrals

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Abstract. Complex function is a function that takes complex numbers as independent and dependent variables. The theory of complex functions originated in the eighteenth century, and in nineteenth century, the new branch of complex functions dominated mathematics. The theory about complex function is called complex analysis, which is very helpful in many branches of mathematics, including algebraic geometry, number theory, analytic combinatorics, applied mathematics as well as Physics, including the branches of hydrodynamics, thermodynamics, quantum mechanics, and twistor theory. This paper is going to introduce an important theorem in complex analysis, and it is the residue theorem. The residue theorem can convert some integrals into integrals with complex variables, which can be solved by using Taylor series and it's an easier method. With the explanation of those examples, the applications of using the theorem to solve functions are clearly shown. This paper is introducing several effective methods of how the residue theorem can be used in practical issues by solving integrals.

Keywords: Residue theorem; Complex analysis; Singularity; Infinite integral.

1. Introduction

Residue theorem is an important theorem in the theory of complex function, it is use to calculate the function's residue at some point. Residue is the special value of a complex function in one isolated singularity and it can be used to calculate the integral value of the function at that point. In the mid-16th century, Italian mathematician Cardin discovered the mathematical idea of complex square roots when solving cubic equations. From the 17th to the 18th century, people tried to explain functions from a geometric level and correspond to each other and be scalar-oriented to solve problems in real life. In the 18th century, the theory of complex variable functions was developed, and in the 19th century it was fully developed. Since the 20th century, complex functions have been widely used in many fields such as elastic physics, theoretical physics, and celestial mechanics [1]. In addition, our country has also achieved fruitful results in mathematical research, especially on the basis of research on complex functions, and was at the world's advanced level at that time. The theory of polycomplex functions was formally established at the end of the 19th century. The research of Hartoggs and Cusin revealed the properties of polycomplex holomorphic functions. In the 1930s, the research on functions of multiple complex variables was officially carried out in an all-round way, and the famous uniqueness theorem of holomorphism was studied. Researchers has also conducted fruitful research on the central problem of multiple complex variables, which has had a profound impact on the subsequent development of functions of multiple complex variables [2].

The residue theorem is a very important theorem, and it is widely used in calculating real function integrals. For the function of real variables that are difficult to solve analytically, this paper can use the residue theorem to solve these problems [3]. The main idea is to transform the real variable function into the complex variable function, and use the residue theorem to calculate and solve the integral. Firstly, this paper transforms the real function into an integral along the closed loop curve. Secondly, it transforms the problems into calculating the residual value at each isolated singularity inside the closed loop. Finally, the residue theorem is used to get the solution of the integrand function for those real variable functions that are difficult to solve analytically. Also, the residue theorem is also important in other subjects such as Physics. By using the residue theorem to get the expression of second-order displacement, the residue theorem finds that the topological phase transition is characterized by this physical quantity [4].

In section 2 this paper describes the different methods of residue theorem and examples to explain it. In section 3 there are three particular questions with the solving process and the explanation of the questions. The last section is for the conclusion.

2. Methods

2.1. Residue and Residue Theorem

Let the analytic function $f(z)$ take an isolated singularity at a. That is, there is a $\rho > 0$ exist to make $U(a,\rho)$ analytic. Let $C_r := \{|z-a| = r\}$, for $0 < r_1 < r_2 < \rho$, $C_{r1} - C_{r2}$ homology is zero with respect to this hollow disk. Hence, for $r \in (0,\rho)$, integral $\mathrm{Res}_{z=a} f(z) := \frac{1}{2\pi i}\int_{C_r} f(z)dz$ is a constant, and can be written as the residue of $f(z)$ at isolated singularity a. Equivalently, there is a residue R of $f(z)$ at isolated singularity a, and it only has original function when $f(z) - \frac{R}{z-a}$ is in $U(a,\rho)$. For residue in pole situation, this paper uses the coefficient of the main part of pole to show $f(z) = \frac{B_k}{(z-a)^k} + \cdots + \frac{B_1}{z-a} + \varphi(z)$ [5]. Therefore, the following equation hold:

$$Res_{z=a} f(z) = \lim_{z\to a} \frac{1}{(n-1)!}(z-a)^n f(z) \tag{1}$$

In the region of $f(z)$, except of the analytic for isolated singularity $\{a_i\}$, one has $\frac{1}{2\pi i}\int_\gamma f(z)dz = \sum_i n(\gamma, a_i) \cdot Res_{z=a_i} f(z)$ for any other close chain γ in the region and any singularities homologous to zero. Write the isolated singularity as $\{a_i\}_{i=1}^n$, use this point as the centre of circle C_i, and require all the circumference only contain this singularity, this can be done by isolation. Remove all the singularities that are homologous to zero about the region, so it is arrived that [6]

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \sum_i n(\gamma, a_i)\frac{1}{2\pi i}\int_{C_i} f(z)dz = \sum_i n(\gamma, a_i) Res_{z=a_i} f(z)\,, \tag{2}$$

and the theorem has been proved.

2.2. Taylor series theorem

Taylor series use infinite term additive, which also called series to calculate a function and those terms that added together is calculated by the derivative of the function at one point. The power series can be defined as [7]

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \tag{3}$$

When $x = 0$, $f(x) = a_0$, differentiate the given function and one can get $f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$. Again, when $x = 0$, $f'(0) = a_1$. So, differentiate it again, this paper get $f''(x) = 2a_2 + 6a_3 x + 12a_4 x^2 + \cdots$. Now, substitute x=0 in second-order differentiation, and get $f''(x) = 2a_2$. Therefore, $[f''(0)/2!] = a_2$. By generalizing the equation, this paper gets that $f^n(0)/n! = a_n$. Now substitute the values in the power series we get,

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \tag{4}$$

Generalize f in more general form, it becomes that

$$f(x) = b + b_1(x-a) + b_2(x-a)^2 + b_3(x-a)^3 + \cdots, \tag{5}$$

with $b_n = f^n(a)/n!$. Now, substitute b_n in a generalized form and this is the formula of taylor series, it is found that

$$f(x) = f(a)\frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots. \quad (6)$$

3. Applications of Residue Theorem in Integrals

3.1. Example I

Calculate the integral [8]

$$I = \int_0^\infty f(x)dx = \int_0^\infty \frac{dx}{1+x^n} \quad (7)$$

when $n = 3$. Residue Theorem can transfer the real integrals into closed loop integrals in complex plane. The integrated function $f(z) = 1/(1+z^3)$ has three simple poles $z_1 = e^{i\pi/3}$, $z_2 = e^{i\pi}$, $z_3 = e^{i5\pi/3}$. In the complex plane and they all on the unit circle.

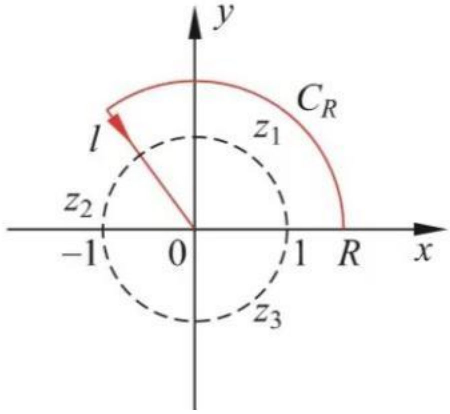

Fig. 1 A closed loop with no singularities on an integral path

Construct a $1/3$ circular arc anticlockwise closed loop as shown in Fig. 1. The loop contains positive real axis, $1/3$ big arc C_R with radius of R, and a ray l with argument principal value of $2\pi/3$. The closed loop only surrounds the simple pole z_1. By using the residue theorem, one can get that the integral of the anticlockwise closed loop is

$$\int_0^R \frac{dx}{1+x^3} + \int_{C_R} \frac{dz}{1+z^3} + \int_l \frac{d\zeta}{1+\zeta^3} = 2\pi i Resf(z_1). \quad (8)$$

When $R \to \infty$, the first integral in Eq. (8) is the integral of infinite integral that this paper will calculate for, the second integral can be proved to be zero, and the third is relative to the desired integral. The Res $f(z_1)$ on the left of function Eq. (8) is the residue of simple pole z_1. In the second integral, let the variate $z = Re^{i\theta}$ and when $R \to \infty$, function $|zf(z)| < \varepsilon \to 0$. Thus, the integral

$$\left|\int_{C_R} f(z)dz\right| = \left|\int_{C_R} zf(z)\frac{dz}{z}\right| \le \int_{C_R} |zf(z)|\frac{|dz|}{|z|} < \int_0^{2\pi/3} \varepsilon\frac{Rd\theta}{R} = \frac{2\pi}{3}\varepsilon \quad (9)$$

which approaches zero as $\varepsilon \to 0$. Let the variable $\zeta = \rho e^{i2\pi/3}$, so the integral changes to

$$\int_l \frac{d\zeta}{1+\zeta^3} = \int_\infty^0 \frac{e^{\frac{i2\pi}{3}}d}{1+\rho^3} = -e^{\frac{i2\pi}{3}}\int_0^\infty \frac{dx}{1+x^3} = -e^{\frac{i2\pi}{3}}I \quad (10)$$

The simple pole residue on the right of Eq. (8) is

$$\text{Res}\, f\,(z_1) = \lim_{z\to z_1}\frac{z-z_1}{1+z^3} = \frac{1}{3z^2}\Big|_{z\,=\,z_1} = \frac{1}{3}e^{-\frac{i2\pi}{3}} = -\frac{1}{3}e^{\frac{i\pi}{3}}. \tag{11}$$

Substitute the above results into Eq. (8), one can get the desired integral

$$I = \int_0^\infty \frac{dx}{1+x^3} = \frac{2\pi i}{1-e^{i2\pi/3}}Resf(z_1) = -\frac{2\pi i}{3}\frac{e^{i\pi/3}}{1-e^{i2\pi/3}} = \frac{\pi}{3}\frac{2i}{e^{i\pi/3}-e^{i\pi/3}} = \frac{\pi}{3}\csc\frac{\pi}{3}. \tag{12}$$

If one chooses the anticlockwise closed loop made from $2/3$ big arc C_{R2} and the ray l_2 with argument principal value $4\pi/3$, the loop will contain two simple poles z_1 and z_2. By using the residue theorem, this paper could get

$$\int_0^\infty \frac{dx}{1+x^3} + \int_{C_{R2}} \frac{dz}{1+z^3} + \int_{l2} \frac{d\zeta}{1+\zeta^3} = 2\pi i[\text{Res}f(z_1) + \text{Res}f(z_2)]. \tag{13}$$

Identically, the second integral on the left of Eq. (11) is 0. Let $\zeta=\rho e^{i4\pi/3}$, the third integral

$$\int_{l2} \frac{d\zeta}{1+\zeta^3} = \int_\infty^0 \frac{e^{i4\pi/3}d\rho}{1+\rho^3} = -e^{i4\pi/3}\int_0^\infty \frac{dx}{1+x^3} = -e^{i4\pi/3}I \tag{14}$$

The residue of singularity z_2 on the right of the Eq. (11) is

$$\text{Res}f(z_2) = \lim_{z\to z_2}\frac{z-z_2}{1+z^3} = \frac{1}{3z^2}\Big|_{z\,=\,z_2} = \frac{1}{3} \tag{15}$$

Substitute the above results into Eq. (11), one can get the desired integral

$$I = \int_0^\infty \frac{dx}{1+x^3} = \frac{2\pi i}{1-e^{i4\pi/3}}[\text{Res}f(z_1) + \text{Res}f(z_2)] = \frac{2\pi i}{3}\frac{e^{-i2\pi/3}+1}{1-e^{i4\pi/3}} = \frac{\pi}{3}\csc\frac{\pi}{3}. \tag{16}$$

The obtained result is same with that get from Eq. (10).

3.2. Example II

Evaluate the integral [9]

$$I = \int_0^\infty \frac{\cos ax + x\sin ax}{1+x^2}dx \tag{17}$$

when $a > 0, a = 0, a < 0$. Firstly, the author writes the integral in form of

$$I = \frac{1}{2}\int_{-\infty}^\infty dx\frac{\cos ax}{1+x^2} - \frac{1}{2}\frac{d}{da}\int_{-\infty}^\infty \frac{\cos ax}{1+x^2}. \tag{18}$$

So, consider the contour integral $\oint_{C(a)} dz\frac{e^{iaz}}{1+z^2}$. Here, $C(a)$ is a semicircle of radius R in the upper half plane when $a > 0$ and in the lower half plane when $a < 0$. For example, when $a > 0$, the integral is

$$I = \int_{-R}^R dx\frac{e^{iax}}{1+x^2} + iR\int_0^\pi d\phi e^{i\phi}\frac{e^{iaR\cos\phi}e^{-aR\sin\phi}}{1+R^2e^{i2\phi}} \tag{19}$$

As $R\to\infty$, the magnitude of the second integral is bounded by

$$\frac{2R}{R^2-1}\int_0^{\pi/2} d\phi e^{-2aR\phi/\pi} = \frac{\pi}{a(R^2-1)}(1-e^{-aR}), \tag{20}$$

which is clearly disappear in this limit. On the other hand, the contour integral is also equal to $i2\pi$ times the residue at the simple pole $z = i$, so that

$$\int_{-\infty}^{\infty} dx \frac{\cos ax}{1+x^2} = i2\pi \frac{e^{-a}}{2i} = \pi e^{-a} \quad (a > 0). \tag{21}$$

For $a < 0$, the author closes in the lower half plane (or simply use the evenness of the cosine) and finds that $\int_{-\infty}^{\infty} dx \frac{cos\, ax}{1+x^2} = \pi e^{-|a|}$. By taking the derivative of this integral (separately for the cases $a > 0$ and $a < 0$), one finds that the original integral is

$$\int_{0}^{\infty} dx \frac{cos\, ax + x\, sin\, ax}{1+x^2} = \frac{\pi}{2}(1 + sgn\, a)e^{-|a|} = \pi\theta(a)e^{-|a|} \tag{22}$$

where θ is the Heaviside step function, which is equal to 1/2 when $a = 0$.

3.3. Example III

Compute the integral [10]

$$I = \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+is)^2} dx\,. \tag{23}$$

Let $f(z) = e^{-z^2/2}$ and define

$$I(s) = \int_{-\infty}^{\infty} f(x + is)dx\,, \tag{24}$$

where $I(s) = I(0)$ for all values of $s \in \mathbb{C}$. This paper can clearly see that $I = (s + it) = I(s)$ for $t \in \mathbb{R}$, so assume that $s \in \mathbb{R}$. Actually, this paper assumes that $s > 0$ as it is clear in the calculation.
For $A > 0$, let

$$I_A(s) = \int_{-A}^{A} f(x + is)dx \tag{25}$$

So $I(s) = \lim_{A\to\infty} I_A(s)$. Make R_A to be the rectangle with vertices $\pm A, (\pm A + is)$. As f is entire, by using the Cauchy's Theorem this paper can shows that $\int_{R_A} f = 0$. Use an obvious method to parametrize R_A and letting $[p, q]$ denote the line segment from p to q, it is precisely that$I_A(0) = I_A(s) = \int_{[-A,-A+is]} f - \int_{[A,A+is]} f$. As $\left|e^{\alpha+i\beta}\right| = e^{\alpha}$, this paper can get $|f(x + iy)| = e^{(y^2-x^2)/2}$. Hence get $|f(z)| \leq e^{(s^2-A^2)/2}$ for $z \in |A, A + is|$. So, $\lim_{A\to} \int_{[A,A+is]} f = 0$ by uniform convergence. Similarly, for $[-A, -A + is]$ a same conclusion is achieved. Hence, this paper gets that $\lim_{A\to\infty} \left(I_A(0) - I_A(s)\right) = 0$. It can also be written as $I(0) = I(s) = 0$.

4. Conclusion

Residue is a key concept in complex function, a function that takes complex numbers as independent and dependent variables. It usually used in some special real integrals and can simplify the calculation process. In this paper, the application of the residue theorem in different types of complex integrals and in practical issues have been shown basically, and it obtains the definition of the complex plane, singularities and the Taylor series. In complex analysis, the residue theorem is a powerful tool to evaluate line integrals of analytic functions over closed curves. Integrals can be computed by finding their singularities on the complex plane and using the residue theorem, and this theorem can transform the real variable function into the complex variable function, this can make the calculation more efficient. Also, for some difficult problems the residue theorem offered a clearer and better calculation method and this theorem facilitate the development of mathematics, computer science as well as physics. In conclusion, with the help of several examples, this paper mentions the high status of residue theorem in different fields such as calculus, and point out the application of residue in simplify the calculate process in different subjects, also, this paper briefly introduced the use of Taylor series in this theorem.

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